Math 227 FINAL EXAM

Do not use any books or notes. You can use a calculator, but not graphing calculator. If you use a calculator, leave your results in exact form instead of decimal form. Show all work for full credit.

1. Find the velocity, acceleration, and speed of a particle with the given position function \( \mathbf{r}(t) = \sqrt{2t} \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k}. \) (5 points)

2. Find the limit, if it exists, or show that the limit does not exist. You need to justify your answer. (15 points)

   (a) \[ \lim_{(x,y) \to (5,-2)} (x^5 + 4x^3y - 5xy^2) \]

   (b) \[ \lim_{(x,y) \to (0,0)} \frac{8x^2y^2}{x^4 + y^4} \]

   (c) \[ \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \]
3. Find the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $\sqrt{20 - 1.95^2 - 7(1.08)^2}$. (8 points)

4. Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$. (14 points)

(a) $z = x^2 + xy + y^2, \quad x = s + t, \quad y = st$
(b) $z = e^x \cos y, \quad x = st, \quad y = \sqrt{s^2 + t^2}$
5. The equation \(xyz = \cos(x + y + z)\) defines \(z\) as a function of \(x\) and \(y\). Find \(\partial z/\partial x\) and \(\partial z/\partial y\). (8 points)

6. Find the maximum rate of change and the direction in which it occurs for \(f(x, y) = \ln(x^2 + y^2)\) at the point \((1, 2)\). (8 points)
7. (22 points)

(a) Evaluate $\iiint_E x^2 \, dV$, where $E$ is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

(b) Use spherical coordinates to evaluate

$$
\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dy \, dx
$$
8. (12 points)

(a) Determine whether or not \( F(x, y) = (2x \cos y - y \cos x) \mathbf{i} + (-x^2 \sin y - \sin x) \mathbf{j} \) is a conservative vector field.

(b) If it is, find a function \( f \) such that \( F = \nabla f \).

(c) Evaluate \( \int_C F \cdot d\mathbf{r} \) along curve \( C \): \( C \) is the upper semicircle that starts at \((0, 1)\) and ends at \((2, 1)\). Can you use your results in (a) and/or (b)?
9. Use Green’s Theorem to evaluate \[ \int_C xy \, dx + 2x^2 \, dy, \] where \( C \) is positively oriented and consists of the line segment from \((-2, 0)\) to \((2, 0)\) and the top half of the circle \( x^2 + y^2 = 4 \). (8 points)

10. **Bonus** (10 points extra) Apply the second vector form of Green’s Theorem to \( \mathbf{F}(x, y) = xi + yj \) and \( C \) given by \( x^2 + y^2 = 4 \), and express \( A \) in terms of \( s \), where \( A \) is the area of the region bounded by \( C \) and \( s \) is the circumference of \( C \).