EGR 265-6D, Math Tools for Engineering Problem Solving
December 14, 2012, 1:30pm to 4:00pm

Name (Print last name first): ...........................................

Student ID Number: .............. ...... ...........

| Problem 1 |          |
| Problem 2 |          |
| Problem 3 |          |
| Problem 4 |          |
| Problem 5 |          |
| Problem 6 |          |
| Problem 7 |          |
| Problem 8 |          |
| Bonus Problem |      |
| Problem 9 |          |
| Problem 10 |         |
| Total |          |
Problem 1 (8 points)

Find an explicit solution of the initial value problem

\[ y' = ye^{2x}, \quad y(0) = 1. \]
Problem 2 (10 points)

Iron, which has a melting point of 2200°F, is heated in a furnace to 2500°F. After extraction from the furnace it cools down according to Newton’s law of cooling in a surrounding medium of temperature 100°F. After two hours the Iron has cooled to 2300°F.

(a) Write down the IVP governing the cooling process using an unknown cooling rate $k$.

(b) Solve the IVP and determine $k$ by using information provided in the problem.

(c) How long does it take the Iron to solidify?

Note: In parts (b) and (c) logarithms do not need to be evaluated.
Problem 3 (12 points)

Consider the second order differential equation

\[ y'' - 4y' + 5y = 2e^{-x}. \]  \hspace{1cm} (1)

(a) Find the general solution of the homogeneous equation corresponding to (1).
(b) Find a particular solution of the inhomogeneous equation (1).
(c) Find the general solution of the inhomogeneous equation (1).
An 8 pound weight stretches an undamped spring by 2 feet. Assume that $g = 32 \text{ ft/s}^2$. Include the correct units in all your answers below.

(a) Find the spring constant $k$ and its correct unit.

(b) Set up the second order differential equation which governs the motion of the spring-mass system, choosing the $x$-axis to be oriented downwards. Find the general solution of this equation.

(c) Find the particular solution of the equation if the mass is released from rest at a position of six inches above the equilibrium.

(d) What is the first positive time at which the mass returns to the equilibrium position?
Problem 5 (10 points)

(a) Find the gradient of $f(x, y) = \frac{1}{x+y}$.
(b) Evaluate the directional derivative of $f(x, y)$ at the point with coordinates $(0, 1)$ in the direction of the vector $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.
(c) Find a unit vector in the direction of steepest increase of $f(x, y)$ at the point $(0, 1)$. 
Problem 6 (8 points)

Find the equation for the tangent plane to the graph of \( z = ye^{x-y} \) at the point \( (1, 1, 1) \).
Problem 7 (10 points)

Find the line integral

$$\int_C 2 \, dx + y\sqrt{1 + 2xdy},$$

where the curve $C$ is given by the graph of $x = \frac{1}{2}y^2$, $0 \leq y \leq \sqrt{3}$. 
Problem 8 (12 points)

(a) Determine for each of the following force fields if it is conservative.

(i) $\mathbf{F}(x, y) = xy\mathbf{i} - xy\mathbf{j}$

(ii) $\mathbf{F}(x, y) = (e^y + ye^x)\mathbf{i} + (xe^y + e^x + 1)\mathbf{j}$

(b) For the conservative force field $\mathbf{F}(x, y)$ from part (a) find a potential function $\phi(x, y)$ and calculate the work done by the force field along the curve $x = t^3, y = 1 - t^2, 0 \leq t \leq 1$. 
Bonus question (8 points)

(c) For the non-conservative force field \( \mathbf{F}(x, y) \) from part (a), use Green’s Theorem to find the work done along the curve \( C \), where \( C \) is the positively oriented triangle with vertices \((0, 0), (1, 1), \) and \((0, 1)\).
Problem 9 (10 points)

A lamina of constant density $\rho(x, y) = 1$ is bounded by the curves $y = x^2$, $x = 0$, and $y = 1$.
(a) Find the moment of inertia $I_y$ with respect to the $y$-axis.
(b) Find the moment of inertia $I_x$ with respect to the $x$-axis.
Problem 10 (8 points)

Rewrite the function \( f(x, y) = \frac{y}{\sqrt{x^2+y^2}} \) using polar coordinates and find its integral over the quarter disk of radius 1 in the first quadrant.