UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN
APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

September, 1996

Instructions: Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. Include all work. Full credit will accrue from answering 5 of the 7 problems given. Indicate which solutions you want to be graded if you work on more than 5 problems.

1. Let $A, B \in \mathbf{C}^{n\times n}$ be given. Recall that the *trace* of a matrix is the sum of the diagonal elements:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

   (a) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Does this also hold for $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times m}$?

   (b) Show that if A and B are similar matrices then they have the same trace.

   (c) Show that the trace of a matrix equals the sum of its eigenvalues (counted up to their algebraic multiplicity).

2. Let $A \in \mathbf{C}^{8\times 8}$ have characteristic polynomial $C_A(x) = (x-3)^8$, minimal polynomial $M_A(x) = (x-3)^4$ and $\dim E_3 = 3$, where E_3 is the eigenspace of A corresponding to the eigenvalue 3. List all the possible Jordan canonical forms for A and give reasons for your answer.

3. Let V be a finite-dimensional inner product space, and let W be a subspace of V. Then $V = W \oplus W^{\perp}$, that is, each $\alpha \in V$ is uniquely expressed in the form $\alpha = \beta + \gamma$ with $\beta \in W$ and $\gamma \in W^{\perp}$. Define a linear operator U by $U\alpha = \beta - \gamma$.

   (a) Prove that U is both self-adjoint and unitary.

   (b) If V is $\mathbf{R}^3$ with the standard inner product and W is the subspace spanned by $[1, 0, 1]^T$, find the matrix representation of U in the standard ordered basis.

4. Consider the system

$$\begin{bmatrix} \epsilon & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   Compute the LU factorization of the coefficient matrix and solve this system, both with and without partial pivoting. Assume that $|\epsilon| \ll 1$, and use the rounding error models (at all stages of the computation!)

$$a + b\epsilon = a$$

(for $a \neq 0$) and

$$a + \frac{b}{\epsilon} = \frac{b}{\epsilon}$$

(for $b \neq 0$). Use this to illustrate why it is desirable to swap rows if the pivot element is non-zero, but small. In particular, compute the product LU for both factorizations, and comment on the results.

5. Let A be a symmetric and positive definite (SPD) matrix. Is A^{-1} an SPD? If so, prove it. If not, explain and give an example.

6. (a) Let $\sigma_1, \dots, \sigma_r$ be the non-zero singular values of a matrix $A \in \mathbf{R}^{m\times n}$. Show that $\sigma_1^2, \dots, \sigma_r^2$ are the non-zero eigenvalues of both $A^T A$ and AA^T.

   (b) Compute the singular values of

$$A = \begin{bmatrix} 0 & -1.6 & 0.6 \\ 0 & 1.2 & 0.8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (c) Let $A \in \mathbf{R}^{n\times n}$ be non-singular. Show that

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_n},$$

   where $\kappa_2(A)$ is the 2-condition number of A, σ_1 and σ_n the largest resp. smallest singular value of A.

7. The matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

   has eigenpairs

$$(\lambda, x) = \left(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \left(-1, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right), \left(3, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$$

   Suppose the power method is applied with starting vector

$$z_0 = [1, 1, -1]^T/\sqrt{3}$$

(a) Determine whether or not the iteration will converge to an eigenpair of A, and if so, which one. Assume exact arithmetic.

(b) Repeat (a), except we now use inverse iteration using the same starting vector z_0 and the Rayleigh quotient of z_0 as approximation for the eigenvalue.

(c) Now answer both (a) and (b) again, except this time use standard fixed precision floating point arithmetic, i.e., computer arithmetic.

Solutions:

1. (a) $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times m}$:

$$\operatorname{tr}(AB) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} = \operatorname{tr}(BA)$$

   (b) $B = QAQ^{-1} \Longrightarrow \operatorname{tr}(B) = \operatorname{tr}(QAQ^{-1}) = \operatorname{tr}(Q^{-1}QA) = \operatorname{tr}(A)$

   (c) use (b) on the Jordan canonical form $J = QAQ^{-1}$ of A

2. 3 is the only eigenvalue. It has algebraic multipicity 8. There are $\dim E_3 = 3$ Jordan chains, the largest one having length $\deg M_A = 4$. Thus only two possibilities are left:

   (i) one block of length 4, one of length 3, one of length 1

   (ii) one block of length 4, two of length 2

3. (a) (i) self-adjoint: need to prove $(Ux, y) = (U^*x, y)$
   Let $x = \beta_1 + \gamma_1, y = \beta_2 + \gamma_2$ where $\beta_1, \beta_2 \in W$, $\gamma_1, \gamma_2 \in W^\perp$
   $(\beta_i, \gamma_j) = 0 \Longrightarrow (Ux, y) = (\beta_1 - \gamma_1, \beta_2 + \gamma_2) = (\beta_1, \beta_2) - (\gamma_1, \gamma_2)$
   $(U^*x, y) = (x, Uy) = (\beta_1 + \gamma_1, \beta_2 - \gamma_2) = (\beta_1, \beta_2) - (\gamma_1, \gamma_2)$
   $\Longrightarrow (Ux, y) = (U^*x, y)$
   (ii) unitary: compute $(U^*Ux, y) = (Ux, Uy) = \ldots = (x, y) \Longrightarrow U^*U = I$

   (b) Result is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

4. If we use the rounding error models that are given to us, then the LU factorizations are (no pivoting)

$$\begin{bmatrix} \epsilon & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/\epsilon & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & -2/\epsilon \end{bmatrix}$$

   and (pivoting)

$$\begin{bmatrix} 2 & 1 \\ \epsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \epsilon/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

The corresponding solutions are (no pivoting)

$$x_1 = 0, x_2 = 1$$

and (pivoting)

$$x_1 = -\frac{1}{2}, x_2 = 1$$

with the product LU being (no pivoting)

$$\begin{bmatrix} 1 & 0 \\ 2/\epsilon & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & -2/\epsilon \end{bmatrix} = \begin{bmatrix} \epsilon & 1 \\ 2 & 0 \end{bmatrix}$$

and (pivoting)

$$\begin{bmatrix} 1 & 0 \\ \epsilon/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ \epsilon & 1 \end{bmatrix}$$

The point of all this is that small divisors can play hell with the accuracy of Gaussian elimination, so it is best to avoid them.

5. The answer to this question is true. Proof is:

   Since $A^T = A$, $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. A^{-1} is symmetric. A is SPD, so $A = LL^T$. Thus, $A^{-1} = (L^{-1})^T L^{-1}$. For any nonzero vector x, the inner product $(x, A^{-1}x)$ is equal to $(x, (L^{-1})^T L^{-1}x) = (L^{-1}x, L^{-1}x)$ which is greater than 0. Therfore A^{-1} is also SPD.

   This problem is simple if a student sees the Cholesky decomposition.

6. (a) This is a simple application of the SVD-Theorem and makes sure that the students know this theorem.

   (b)

   $$A^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   $\implies \sigma_1^2 = 4, \sigma_2^2 = 1, \sigma_3^2 = 0$, i.e. $\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 0$

   (c) A non-singular $\implies \sigma_1 \geq \ldots \geq \sigma_n > 0$
   Eigenvalues of $A^T A$: $\sigma_1^2 \geq \ldots \geq \sigma_n^2 > 0$
   General properties of 2-condition number $\implies \kappa_2(A)^2 = \kappa_2(A^T A) = \sigma_1^2/\sigma_n^2$

7. The issue here is whether or not the student understands the power method(s) and computer arithmetic. The given starting vector is orthogonal to the dominant eigenvector, therefore the power iteration in exact arithmetic will converge to the second dominant eigenvector, $(1, 0, 0)^T$. When we use inverse iteration, we will converge to the least dominant eigenvector, $(0, 1, -1)^T$. When computer arithmetic is used, the rounding error destroys the perfect orthogonality of the starting vector and the dominant eigenvector, hence the direct iteration converges to $(0, 1, 1)^T$; the inverse iteration performs as before.