UNIVERSITY OF ALABAMA SYSTEM

JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS

JOINT PROGRAM EXAMINATION

May, 1998

Linear Algebra and Numerical Linear Algebra

Instructions: *Do any 7 of the 8 problems given. Be sure to indicate which 7 are to be graded. Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems, each of which is "half-correct".*

Notation: $\kappa_\alpha(M)$ indicates the α-norm condition number of a matrix M.

1. Consider the vectors in $\mathbb{R}^4$ defined by $v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 9 \end{bmatrix}$.

   Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

2. Let V be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ which have degree less than or equal to $n-1$. Let $t_1, t_2, \ldots, t_n$ be any n distinct real numbers, and define linear functionals $L_i(p) = p(t_i)$ on V. Show that $L_1, L_2, \ldots, L_n$ are linearly independent.

3. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. Show that $\min\{\frac{\|\delta A\|_2}{\|A\|_2} \mid A + \delta A \text{ is singular}\} = 1/\kappa_2(A)$. (That is the distance to the nearest singular matrix is $1/\kappa_2(A)$.)

4. Compute the condition numbers $\kappa_1(A)$, $\kappa_2(A)$ and $\kappa_\infty(A)$ for

$$A_n = \begin{bmatrix} 1 & 1 - \frac{1}{n} \\ 1 + \frac{1}{n} & 1 \end{bmatrix}$$

   where $n \geq 2$.

5. Given the data $(0,1), (3,4), (6,5)$. Use a QR factorization technique to find the best least squares fit by a linear function.

6. Let A be an $n \times n$ real matrix of full rank and let X be a matrix that diagonalizes A, i.e. $X^{-1}AX = D$ where D is a diagonal matrix with eigenvalues $\lambda_i, i = 1, \ldots, n$ of A on the diagonal. If $A' = A + E$ and λ' is an eigenvalue of A', prove that

$$\min_{1 \leq i \leq n} |\lambda' - \lambda_i| \leq \kappa_2(X) \|E\|_2.$$

7. (a) Let A be a 16×16 complex matrix whose characteristic polynomial and minimal polynomial are $C_A(x) = x^8(x - 2i)^8$ and $M_A(x) = x^5(x - 2i)^5$ respectively. Also let $\dim E_0 = 2$, $\dim E_{2i} = 3$, where E_λ is an eigenspace corresponding to the eigenvalue λ of A. Find a Jordan canonical form of A.

(b) Let A be a 8×8 complex matrix with $C_A(x) = (x^2+1)^4$, $\dim E_i = 1$, and $\dim E_{-i} = 3$. Find the minimal polynomial of A.

8. For any matrix $A = (a_{ij})$, $A \in \mathbb{R}^{n \times m}$ define

$$||A||_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2 \right)^{1/2}$$

This is the Frobenius matrix norm. Show that if $Q \in \mathbb{R}^{n \times n}$ is orthogonal, then $||QA||_F = ||A||_F$. Then show that

$$||A||_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2}$$

where σ_i are the singular values of A.