UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED MATHEMATICS
JOINT PROGRAM EXAMINATION
September, 2007

Linear Algebra and Numerical Linear Algebra

Instructions: Do 7 of the 8 problems given. Completeness in answers is very important. Justify your steps by referring to theorems by name where appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems, each of which is "half-correct".

1. Let F be a field and $A \in F^{n\times n}$ have rank 1. Prove:

(a) $A^2 = cA$ for some $c \in F$,

(b) A is similar to a diagonal matrix if and only if $c \neq 0$.

2. Let $A = (a_{kj})$ be a complex $n \times n$ matrix.

(a) Assume that the inner product $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{C}^n$. Prove that $A = 0$.

(b) Find a real $n \times n$ matrix $A \neq 0$ such that $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^n$.

3. Let v, x and y be nonzero vectors and let v^{H} denote the conjugate transpose of v.

(a) Prove that $U = I - vv^{\mathrm{H}}$ is unitary if and only if $\|v\|_2 = \sqrt{2}$.

(b) Prove that if $\|x\|_2 = \|y\|_2$ and if the inner product of x and y is real, then there exists a unitary matrix U of the form $I - vv^{\mathrm{H}}$ such that $Ux = y$ for some vector v.

(c) Exploit the above results to find a QR factorization of the matrix A given below, such that $A = QR$ where Q is unitary and R is an upper triangular matrix.

$$A = \begin{bmatrix} 63 & 41 & -88 \\ 42 & 60 & 51 \\ 0 & -28 & 56 \\ 126 & 82 & -71 \end{bmatrix}.$$

4. Solve the system

$$\begin{bmatrix} 0.001 & 1.00 \\ 2.00 & 3.00 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.00 \\ 5.00 \end{bmatrix}$$

by the LU decomposition with and without partial pivoting, using chopped arithmetic with a three digit mantissa. Obtain computed solutions (x_c, y_c) in both cases. Find the exact solution, compare, make comments.

5. The spectral radius of $A \in \mathbb{C}^{n\times n}$ is defined by

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ an eigenvalue of } A\}.$$

Show that

(a) $\rho(A) \leq \|A\|$ for every matrix norm $\|\cdot\|$ that is induced by a norm on $\mathbb{C}^n$;

(b) if A is normal then $\|A\|_2 = \rho(A)$.

6. Prove the following

(a) Suppose that $M \in \mathbb{C}^{2\times 2}$ has all entries real and two complex conjugate eigenvalues $a \pm ib$ (where $a, b \in \mathbb{R}$ and $b \neq 0$). Show that there exists a nonsingular $Q \in \mathbb{R}^{2\times 2}$ such that $Q^{-1}MQ = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

(b) Prove: If $A \in \mathbb{R}^{n\times n}$ then there exists a nonsingular matrix $P \in \mathbb{R}^{n\times n}$ such that $P^{-1}AP$ is a block upper triangular matrix with each of its diagonal blocks being square of order 1 or 2.

7. $A, B \in \mathbb{C}^{n\times n}$ are simultaneously similar to upper triangular matrices if both $P^{-1}AP$ and $P^{-1}BP$ are upper triangular for some nonsingular $P \in \mathbb{C}^{n\times n}$.

(a) Let A and B be simultaneously similar to upper triangular matrices. Prove that zero is the only eigenvalue of the matrix $AB - BA$.

(b) Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. If possible use the result in part (a) to determine whether these two matrices are simultaneously similar to upper triangular matrices.

8. A matrix A is normal if $AA^{\mathrm{H}} = A^{\mathrm{H}}A$, where A^{H} is the conjugate transpose of A. Prove

(a) If A is a normal matrix then A and A^{H} have the same eigenvectors.

(b) If A is a normal matrix and two vectors x and y are eigenvectors of A corresponding to different eigenvalues, then the vectors x and y are orthogonal.

(c) If A is a normal and upper triangular matrix then A is diagonal.