UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

May 2012

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen

1. Let $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ matrices over the real field $\mathbb{R}$. Let $A \in \mathbb{R}^{n\times n}$ have singular value $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Suppose that $\|A\|_2 = \sigma_n$. Compute the 2-norm condition number $\kappa_2(A)$. What else can you say about A?

2. Let $\mathbb{C}^{n\times n}$ denote the set of $n \times n$ matrices over the complex field $\mathbb{C}$.

   (a) Show that if $B \in \mathbb{C}^{n\times n}$ is Hermitian and $x^*Bx = 0$ for all $x \in \mathbb{C}^n$, then $B = 0$.

   (b) Show that for any $A \in \mathbb{C}^{n\times n}$ there is a unique ordered pair of Hermitian matrices $B, C \in \mathbb{C}^{n\times n}$ for which $A = B + iC$ (here $i = \sqrt{-1}$).

   (c) Let $A \in \mathbb{C}^{n\times n}$. Show that if x^*Ax is real for all $x \in \mathbb{C}^n$, then A is Hermitian.

3. Suppose $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and $x, b \in \mathbb{R}^n$. Let $\tilde{A}$ be a perturbed matrix and $\tilde{x}$ be the corresponding solution of the perturbed system, i.e., $\tilde{A}\tilde{x} = b$. Show that

$$\frac{\|x - \tilde{x}\|}{\|\tilde{x}\|} \leq \kappa(A) \frac{\|A - \tilde{A}\|}{\|A\|}$$

   where the condition number $\kappa(A)$ is defined with respect to the same norm $\|\cdot\|$ as appears in the above inequality.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Define a linear transformation $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(B) = AB - BA$.

   (a) Choose an ordered basis of $\mathbb{R}^{2\times 2}$ and compute the matrix that represents T with respect to that basis.

   (b) Find a basis for each of the eigenspaces of T.

   (c) Give the minimal polynomial of T and the Jordan form for T.

5. Let $A \in \mathbb{R}^{m\times n}$ for $m > n$ be a full rank matrix and $A = QR$ its QR decomposition, where $Q \in \mathbb{R}^{m\times m}$ is an orthogonal matrix and $R \in \mathbb{R}^{m\times n}$ is upper triangular. Let $R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$, where $\hat{R} \in \mathbb{R}^{n\times n}$ is a square upper triangular matrix. Show that the solution of the least

squares problem $Ax \approx b$ can be found by solving $\hat{R}x = \hat{c}$, where $\hat{c} \in \mathbb{R}^n$ consists of the first n components of the vector $c = Q^T b$, i.e., $c = \begin{bmatrix} \hat{c} \\ d \end{bmatrix}$, where $d \in \mathbb{R}^{m-n}$.

6. Suppose $A, B \in \mathbb{C}^{m \times m}$ are two matrices with no common eigenvalues, i.e., Spectrum$(A) \cap$ Spectrum$(B) = \emptyset$.

   (a) Let $p_A(x)$ denote the characteristic polynomial of A. Prove that the matrix $p_A(B)$ is nonsingular.

   (b) Let $X \in \mathbb{C}^{m \times m}$ satisfy $AX = XB$. Prove that for any polynomial $P(x)$ we have $P(A)X = XP(B)$. Then prove that $X = 0$.

   (c) Prove that for every $C \in \mathbb{C}^{m \times m}$ there exists a unique $X \in \mathbb{C}^{m \times m}$ such that $AX - XB = C$.

7. Let $A \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix, i.e., $A^T = -A$.

   (a) Show that if A has a real eigenvalue $\lambda \in \mathbb{R}$, then $\lambda = 0$;

   (b) Show that for any real number $c \neq 0$ the matrix $A + cI$ is invertible.

   (c) Suppose $M \in \mathbb{R}^{n \times n}$ is a matrix consisting of zeros and ones such that $M_{ii} = 0$ for all $i = 1, \ldots, n$ and $M_{ij} = 0$ if and only if $M_{ji} = 1$ for all $i \neq j$. Show that $M = A + B$, where A is skew-symmetric and B is symmetric. Describe the matrix B.

8. Assume the matrices of this problem are all $n \times n$ with real entries.

   (a) Show that for arbitrary matrices A and B

   $$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

   (b) Let A be arbitrary and J be the matrix whose all components are equal to 1, i.e., $J_{ij} = 1$ for all $1 \leq i, j \leq n$. Show that

   $$\operatorname{rank}(A) \geq \operatorname{rank}(A + cJ) - 1$$

   for all $c \in \mathbb{R}$.

   (c) Suppose M is a matrix consisting of zeros and ones such that $M_{ii} = 0$ for all $i = 1, \ldots, n$ and $M_{ij} = 0$ if and only if $M_{ji} = 1$ for all $i \neq j$. Show that $\operatorname{rank}(M) \geq n - 1$.