UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam in Linear Algebra and Numerical Linear Algebra

May, 2013

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let V and W be vector spaces with $\dim V = \dim W < \infty$. Let $T \in L(V, W)$ and $S \in L(W, V)$ satisfy $(ST)(x) = x$ for every $x \in V$. Without assuming T or S being invertible establish the following:

   (a) T is one-to-one;

   (b) T is onto;

   (c) T^{-1} exists and $T^{-1} = S$ (i.e., $ST = I_V$ and $TS = I_W$, where I_V and I_W are the identity maps on V and on W, respectively);

   (d) If A and B are square matrices with $AB = I$, then both A and B are invertible and $A^{-1} = B$, $B^{-1} = A$.

2. Let $A \in \mathbb{C}^{n \times n}$ satisfy $A^2 = A$. Prove that

$$\operatorname{rank}(A) + \operatorname{rank}(I - A) = n,$$

   where $I \in \mathbb{C}^{n \times n}$ is the identity matrix.

3. Let V and W be finite dimensional vector spaces, and let $T \in L(V, W)$ be a linear transformation of rank r where $1 \leq r < \min\{\dim(V), \dim(W)\}$. Prove that there exist bases α for V and β for W such that the matrix representation for T with respect to α and β has the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

   where I_r is the $r \times r$ identity matrix.

4. (a) Let A be a 10×10 complex matrix with characteristic polynomial $C_A(x) = (x-1)^6(x+2)^4$, minimal polynomial $M_A(x) = (x-1)^3(x+2)^2$, and $\dim E_1 = 3$, $\dim E_{-2} = 2$, where E_1 and E_{-2} are the eigenspaces corresponding to the eigenvalues 1 and -2, respectively. Find a Jordan canonical form of A.

   (b) Let A be an 8×8 complex matrix with characteristic polynomial $C_A(x) = (x+i)^3(x-i)^3(x-1)^2$, $\dim E_{-i} = \dim E_i = \dim E_1 = 2$. Find the minimal polynomial of A.

5. Let $Q, Z \in \mathbb{R}^{n \times n}$ be orthogonal matrices. For any matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ show the following:

   (a) $\|QAZ\|_2 = \|A\|_2$,

   (b) $\|QAZ\|_F = \|A\|_F$, where $||A||_F = \left(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2\right)^{1/2}$ is the Frobenius matrix norm.

6. (a) Let

$$x = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \qquad y = \begin{bmatrix} -4 \\ 4 \\ 4 \\ 0 \\ -4 \end{bmatrix}.$$

Is there an orthogonal matrix Q so that $Qx = y$? If so, use EXACT arithmetic to find it. If not, explain why.

(b) Let

$$A = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}.$$

Compute a QR decomposition of A using Householder reflections and then solve the least square problem $\min_x \|b - Ax\|_2$ using the QR decomposition.

7. (a) Let $A \in \mathbb{C}^{n\times n}$. Show that A is normal (i.e., $AA^* = A^*A$) if and only if there is a unitary matrix V such that $A = A^*V$.

(b) Assume that A is normal. Show that all the eigenvalues of A are purely imaginary if and only if A is skew-Hermitian, i.e., $A^* = -A$.

8. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Show that

$$\min \left\{ \frac{||A - A_s||_2}{||A||_2} : A_s \text{ is singular} \right\} = \frac{1}{\kappa_2(A)}.$$

That is, the relative distance to the nearest singular matrix is $1/\kappa_2(A)$. Here $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$ is the 2-norm condition number of A.