University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 2019

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving statements, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name).
  Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ defined by

$$T(a,b,c,d,e) = (2a, 2b, 2c+d, a+2d, b+2e)\,.$$

   (a) Find the characteristic and minimal polynomial of T.

   (b) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.

   (c) Find the Jordan form of T with respect to your basis.

2. (a) For which value(s) of x are the matrices,

$$A = \begin{bmatrix} 1 & x & 1 \\ x & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

   similar?

   (b) Prove that whenever these two matrices are similar, then they are orthogonally equivalent.

3. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Let λ_1 be the maximum of the eigenvalues of B. For $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$, using the usual 2-norm $\|\mathbf{x}\|_2$, define the Raleigh quotient $\rho_B(\mathbf{x})$ for B by

$$\rho_B(\mathbf{x}) = \frac{(B\mathbf{x},\mathbf{x})}{(\mathbf{x},\mathbf{x})} = \frac{\mathbf{x}^t B\mathbf{x}}{\|\mathbf{x}\|_2^2}$$

   Prove the following:

   (a) If B and λ_1 are defined as above, prove that $\lambda_1 = \max\{\rho_B(\mathbf{x}) :\ \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_2 = 1\}$

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a matrix with largest singular value σ_1. If

$$\|A\|_2 = \max\{\|A\mathbf{x}\|_2 :\ \mathbf{x} \in \mathbb{R}^n \text{ and } \|\mathbf{x}\|_2 = 1\}$$

   show that $\|A\|_2 = \sigma_1$.

4. Let V be an inner product space and $W \subset V$ a finite dimensional subspace with orthonormal basis $\{u_1, \ldots, u_n\}$. For every $x \in V$, define

$$P(x) = \sum_{i=1}^{n} < x, u_i > u_i\,.$$

   (a) Prove that $x - P(x) \in W^{\perp}$, hence P is the orthogonal projection onto W.

   (b) Prove that $\|x - P(x)\| \leq \|x - z\|$ for every $z \in W$, and that if $\|x - P(x)\| = \|x - z\|$ for some $z \in W$, then $z = P(x)$.

5. Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}, n \geq m$, and suppose that A has full rank. Show that $A^T A$ is nonsingular, and the least square problem for the overdetermined system $Ax = b$ has a unique solution.

6. Suppose $\mathbf{A} \in \mathbb{R}^{n\times m}$ has full rank, that is, $rank(\mathbf{A}) = r = min(m,n)$. Let $\sigma_1 \geq ... \geq \sigma_r$ be the singular values of $\mathbf{A}$. Show that $\sigma_r = \min\{\|A - C\|_2 : \text{rank}(C) \leq r-1\}$.

7. Let $\mathbf{A} \in \mathbb{R}^{n\times m}, n \geq m$, and have full rank. Show that $\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$ has a solution where x minimizes $\|Ax - b\|_2$.

8. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show the following:

   (a) A is positive definite if and only if $\det(A_k) > 0$ for $k = 1, \cdots, n$ where A_k is the kth principal minor of A

   (b) Let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and A' be the $k \times k$ matrix formed by the intersections of the rows and columns of A with numbers $i_i, \cdots, i_k$. Then $\det A' > 0$.