University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

May, 2021

- This is a closed book exam. The duration of the exam is **three and one half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points.
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving statements, and give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name). Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. If $Q \in \mathbb{R}^{n\times n}$ satisfies $\|Qx\|_2 = \|x\|_2$ for any $x \in \mathbb{R}^n$, prove that Q is orthogonal.

2. Let $M_2(\mathbb{R})$ denote the set of 2×2 real matrices and

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}.$$

Define the linear mapping $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T(B) = AB - BA.$$

   (i) Fix an ordered basis B of $M_2(\mathbb{R})$ and compute the matrix $[T]_B \in \mathbb{R}^{4\times 4}$ that represents T with respect to this basis.

   (ii) Give the eigenvalues of T.

   (iii) Compute a basis for each of the eigenspaces of T.

   (iv) Give the minimal and characteristic polynomials of T and the Jordan form for T. Say whether T is diagonalizable or not.

3. Define $\mathbb{R}^{n\times n}$ to be the space of all real n-by-n matrices, suppose $S \in \mathbb{R}^{n\times n}$ and define the linear mapping $T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by

$$T(P) = PS + SP.$$

   (i) Prove that if λ is an eigenvalue of S, u is the corresponding eigenvector, $u \in \text{null}\,(T(P))$ and $Pu \neq 0$, then Pu is also an eigenvector of S, with eigenvalue $-\lambda$.

   (ii) Prove that if S is symmetric positive definite, then the mapping T is injective.

4. Suppose that $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is symmetric and semi-positive definite, and that $\sum_{j=1}^{n} a_{ij} = 0$, for $1 \leq i \leq n$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of A, and set $\mathbf{1} = [1, ..., 1]^T \in \mathbb{R}^n$ ($\mathbf{1}$ is the "all ones" vector).

   (i) Show that $\mathbf{1}$ is an eigenvector of A.

   (ii) Prove that $A - \lambda_{n-1}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T)$ is symmetric and semi-positive definite.

5. Let V be a finite dimensional inner product space over $\mathbb{C}$. Let $T : V \to V$ be a self- adjoint operator on V. Suppose $\mu \in \mathbb{C}$, $\epsilon > 0$ are given and assume there is a unit vector $x \in V$ such that

$$\|T(x) - \mu x\| \leq \epsilon\,.$$

Show that there is an eigenvalue λ of T such that

$$|\lambda - \mu| \leq \epsilon\,.$$

6. Let $\delta = 10^{-6}$ and consider the over determined system $Ax = b$ given as

$$\begin{bmatrix} 1 & -1 \\ 0 & \delta \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ \delta \\ 1 \end{bmatrix}.$$

   (i) Determine, by hand, the exact least squares solution to this overdetermined system using the normal equations.

   (ii) If you compute the least squares solution to $Ax = b$ using the normal equations on a computer with machine precision $\mathbf{u} = 10^{-10}$, what result would you expect.

   (iii) The ∞-norm condition number of a matrix A is defined to be

$$\kappa_\infty = \|A\|_\infty \|A^{-1}\|_\infty.$$

   Compute the ∞-norm condition number of the coefficient matrix in the normal equations. Comment on the stability when using the normal equations for solving this least squares problem on the computer described in (ii).

7. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{pmatrix}$$

   (i) Find the reduced QR factorization of A by the Gram-Schmidt process.

   (ii) Use the QR factorization from (i) to find the least squares fit by a linear function for the data points $(2, 3)$, $(3, 4)$, and $(1, 0)$.

   (iii) Find the orthogonal projector on the column space of A, in terms of the orthogonal factor for A, without using A itself.

8. Suppose that $A \in \mathbb{R}^{n\times m}$ has full rank, that is, $r = \operatorname{rank}(A) = \min\{m, n\}$. Let $\sigma_1 \geq ... \geq \sigma_r$ be the singular values of A, and let $B \in \mathbb{R}^{n\times m}$ satisfy $\|A - B\|_2 < \sigma_r$. Prove that B also has full rank.