UNIVERSITY OF ALABAMA SYSTEM

Joint Doctoral Program in Applied Mathematics

Joint Program Exam in Linear Algebra and Numerical Linear Algebra

September 2021

Exam Rules:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. **You are required to do seven of the eight problems for full credit.** If you answer all eight problems, your best seven will be graded.
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let X and Y be finite dimensional vector spaces, and let $T \in L(X, Y)$ be a linear transformation of rank r where $1 \leq r < \min\{\dim(X), \dim(Y)\}$. Prove that there exist bases α for X and β for Y such that the matrix representation for T with respect to α and β has the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix.

2. Let $A, B \in C^{n\times n}$ be given. Recall that the trace of a matrix is the sum of the diagonal elements: if $A = (a_{ij})$, $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

   (a) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Does this also hold for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$?

   (b) Show that if A and B are similar matrices then they have the same trace.

   (c) Show that the trace of a matrix equals the sum of its eigenvalues (counted up to their algebraic multiplicity).

3. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, $b \in \mathbb{R}^n$, and let $x \in \mathbb{R}^n$ denote the exact solution of the linear system $Ax = b$. Assume also that

$$(A + \delta A)y = b + \delta b,$$

and

$$\frac{\|\delta A\|}{\|A\|} \leq \epsilon, \quad \frac{\|\delta b\|}{\|b\|} \leq \epsilon,$$

and $\epsilon\kappa(A) = r < 1$, where $\kappa(A)$ is the condition number of A. Show that $A + \delta A$ is invertible and

$$\frac{\|y\|}{\|x\|} \leq \frac{1+r}{1-r}.$$

4. Consider the linear least squares problem

$$\min_x \|b - Ax\|_2, \quad A = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \; b = \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}. \tag{1}$$

(a) Compute the QR factorization of A using Householder reflections.

(b) Solve the problem (1) using the QR factorization method.

(c) Find the pseudo-inverse of A.

5. For the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1/2 & \sqrt{3}/2 \\ \sqrt{3} & 1 \end{pmatrix},$$

obtain the singular value decomposition of A in the form $A = U\Sigma V^T$ where U and V are orthogonal and Σ is diagonal. Use this to find the Frobenius norm $\|A\|_F$ and the 2-norm $\|A\|_2$ of A.

6. (a) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Determine a similarity transformation $A = PJP^{-1}$, where J is a Jordan form of A.

(b) Let $L(x) = Ax$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find a representation of L in the coordinate system E of eigenvectors of A. You must prove any theorem you wish to use here.

7. Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Prove or disprove: the eigenvalues of $AB - BA$ are purely imaginary. What can you say if $A, B \in \mathbb{C}^{n\times n}$?

8. Let $A, Q_0 \in \mathbb{R}^{m\times m}$. Define sequences of matrices Z_k, Q_k and R_k by

$$Z_k = AQ_{k-1}, \quad Q_kR_k = Z_k, \qquad k = 1, 2, \cdots,$$

where Q_kR_k is a QR factorization of Z_k. Suppose $\lim_{k\to\infty} R_k = R_\infty$ exists.

(a) Does the limit $\lim_{k\to\infty} Q_k = Q_\infty$ necessarily exist? Justify your answer.

(b) Determine the eigenvalues of A in terms of R_∞ if $\lim_{k\to\infty} Q_k = Q_\infty$ exists.