University of Alabama System

Joint Ph.D. Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 8, 2022

- This is a closed book exam. The duration of the exam is **three and an half hours**.
- You are required to do **7 out of the 8 problems** for full credit.
- Each problem is worth 10 points; multiple parts of a given problem have equal weights (unless otherwise specified).
- You must justify your solutions: cite theorems that you use, provide counter examples for disproving theorems, give explanations and show all the calculations for the numerical problems.
- Start each solution on a new page. Write the last four digits of your university **student ID number** and the problem number on every page (do not put your name).
  Write only on one side of the page.
- No calculators are allowed. No other electronic devices are allowed.
- Please write legibly with a pen or a dark pencil.

1. Define $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ by $T(f(x) = xf(x) + f'(x)$.

   (a) Prove that T is a linear operator.

   (b) Find basis β for $\mathrm{N}(T)$.

   (c) Find basis γ for $\mathrm{R}(T)$.

   (d) Compute the nullity and rank of T and verify the dimension theorem.

   (e) Is T $1-$ to -1 or onto? Justify your answer.

2. (a) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Determine a similarity transformation $A = PJP^{-1}$, where J is a Jordan form of A.

   (b) Let $L(x) = Ax$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find a representation of L in the coordinate system E of eigenvectors of A. You must prove anything you want to use.

3. Consider a least squares problem
$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix},$$

   (a) Compute a QR decomposition of the matrix, with exact arithmetic, by using the Householder reflector method.

   (b) Compute the least squares solution based on the QR decomposition of part (a).

4. Show that if $\mathbf{A}$ is SPD then for any $k \times k$ matrix $\mathbf{B}$ of rank k, the matrix $\mathbf{B}^T\mathbf{A}\mathbf{B}$ is SPD.

5. Let
$$A = \begin{bmatrix} 2 & 2 & 0 & 1 & -1 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix}$$

   (a) Find the singular values and left singular vectors (columns of U) of the 2×5 matrix A shown on the right.

   (b) Using your answer to (a) find the 2-norm of the 5×5 matrix A^TA?

   (c) Find the first right singular vector v_1 (1st column of V in SVD of A).

   (d) Find the matrix that miniminizes $\|A - B\|_2$ over all matrices B of rank 1?

6. Let A, B be two $m \times p$ matrices. We wish to find a unitary matrix Q ($Q \in \mathbf{R}^{p\times p}$, $Q^TQ = I$), that minimizes $\|A - BQ\|_F$.
   (a) For $X = A - BQ$, expand $\|X\|_F^2$ as $\mathrm{tr}(X^TX)$ to show that the problem is equivalent to maximizing the trace of A^TBQ.
   (b) Use the SVD of A^TB to find the optimal solution. [Hint: exploit the identity $tr(AB) = tr(BA)$ which is valid when both products exist].

7. Let $V \subset \mathbb{C}^n$ be a k-dimensional vector subspace, $k < n$. Let $\{q_1, q_2, \cdots, q_k\}$ be an orthonormal basis in V. Let $Q \in \mathbb{C}^{n\times k}$ denote the matrix whose columns are $q_1, q_2, \cdots, q_k$. Prove that
$$P = QQ^*$$
   is the orthogonal projection of $\mathbb{C}^n$ onto V, i.e., the projection onto V along $V^{\perp}$.

8. Let $\delta = 10^{-6}$ and consider the overdetermined system $Ax = b$,

$$\begin{bmatrix} 1 & -1 \\ 0 & \delta \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ \delta \\ 1 \end{bmatrix}.$$

(a) Determine, by hand, the exact least squares solution to this overdetermined system using the normal equations.
(b) If you compute the least squares solution to $Ax = b$ via the normal equations on a computer with machine precision $\mathbf{u}= 10^{-10}$, what result would you expect?
(c) The ∞-norm condition number of a matrix A is defined to be

$$\kappa_\infty = \|A\|_\infty \|A^{-1}\|_\infty.$$

Compute the ∞-norm condition number of the coefficient matrix in the normal equations. Comment on the stability if one utilizes the normal equations for solving this least squares problem on the computer described in part (b).