Joint Program Exam of May 2000 in

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test $a.e.$ and dx refer to Lebesgue measure on $\mathbb{R}$.

Part 1.

DO 5 PROBLEMS IN PART ONE. MARK THE ONES TO BE GRADED.

For each of the following statements decide whether it is true or false. Give a succinct proof of your assertion. Sometimes an example may be sufficient.

1) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-x} & x \in \mathbb{Q} \\ e^{x} & x \notin \mathbb{Q} \end{cases}$$

is measurable.

2) If the boundary of $\Omega \subset \mathbb{R}^k$ has outer measure zero, then Ω is measurable.

3) The union of two non-measurable subsets of $\mathbb{R}$ is never measurable.

4) If $f_n \in L^1(\mathbb{R})$, $f_n(x) \leq e^{-|x|}$ and $f_n(x) \to 0$ for almost every x then $\limsup \int f_n \, dx \leq 1$.

5) For every finite positive measure μ on $\mathbb{R}$ there exists a non-negative measurable function f such that for all measurable sets E

$$\mu(E) = \int_E f \, dx.$$

6) There exist two sequences $(a_n) \in l^1$ and $(b_n) \in l^2$ such that $(a_n + b_n)$ is neither in l^1 nor in l^2.

Part 2.

DO 5 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1) Suppose $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and prove that:
(a) f is of bounded variation.
(b) $|f|$ is absolutely continuous and

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)| \qquad a.e.$$

2) It is easy to guess the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx.$$

Prove that your guess is correct. Be sure to justify all steps.

3) Find a function $f \in L^1(\mathbb{R})$ which does not belong to any $L^p(\mathbb{R})$ with $p > 1$.

4) Suppose $f(x)$ and $xf(x)$ belong to $L^1(\mathbb{R})$. Prove that

$$\hat{f}(k) = \int e^{ikx} f(x)\, dx$$

is differentiable and that

$$\frac{d}{dk}\hat{f}(k) = \int e^{ikx} ixf(x)\, dx.$$

5) Let $(a_n)_{n\geq 0}$ be a sequence of non-negative real numbers and for each $t \geq 0$ let $N(t) = \#\{n : a_n > t\}$. That is, $N(t)$ is the number of integers n for which $a_n > t$. Show that

$$\sum_{n\geq 0} a_n = \int_0^\infty N(t) dt.$$

Hint: If a_n is considered as the area of a suitable rectangle in $\mathbb{R}^2$ then the left side becomes an integral over $\mathbb{R}^2$.

6) While preparing for his class, Prof. N. read the following definition of the Lebesgue integral $\int_E f d\mu$ of a bounded measurable function f over a measurable set E of finite measure.

"Let f_k be any sequence of measurable simple functions on E that converges uniformly to f (a function being simple if and only if it achieves only finitely many values). Then $\int_E f d\mu \overset{def}{=} \lim_{k\to\infty} \int_E f_k d\mu$ (for simple functions the integral has already been defined)."

Being shaky on the concept of uniform convergence, Prof. N. decided to simplify the definition by dropping the word "uniform" from it. Which bounded measurable functions have an integral according to Prof. N.'s definition? Prove your assertion.