Joint Program Exam, September 2001

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

For each problem which you attempt try to give a complete solution. A correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Throughout this test m denotes Lebesgue measure on $\mathbb{R}$ and 'measurable' is short for 'Lebesgue-measurable'. Instead of dm we write dx.

1. (a) Suppose that $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ and $A \times B$ is measurable in $\mathbb{R}^2$. Does this imply that A and B are measurable in $\mathbb{R}$?

(b) Does there exist a non-measurable $A \subset [0,1]$ such that $B := \{(x,0) \in \mathbb{R} : x \in A\}$ is a closed subset of $\mathbb{R}^2$?

2. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if x is rational} \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is measurable.
(ii) Prove or disprove that f is of bounded variation on [0, 1].
(iii) Is f Lebesgue integrable, but not Riemann integrable? Justify your answer.

3. Let $E \subset \mathbb{R}$ be measurable and $c \in (0,1)$.

(a) Suppose that $m(E \cap I) \le cm(I)$ for all intervals I in $\mathbb{R}$. Show that $m(E) = 0$.

(b) What can be said about E if $m(E \cap I) \ge cm(I)$ for all intervals I?

4. (a) Let $f : [a,b] \to \mathbb{C}$ and $g : [a,b] \to \mathbb{C}$ be absolutely continuous. Show that fg is absolutely continuous and that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{for a.e. } x \in [a,b].$$

(b) Let g be absolutely continuous on $[0,\pi]$. Use part (a) to show that

$$\lim_{k\to\infty} \int_0^\pi \sin(kx) g(x)\, dx = 0.$$

5. Let $f : \mathbb{R} \to [0,\infty)$ be measurable. Prove that

(a) $E := \{(x,t) \in \mathbb{R} \times [0,\infty) : f(x) > t\}$ is measurable in $\mathbb{R} \times [0,\infty)$,

(b)

$$\int_{\mathbb{R}} f(x)\, dx = \int_0^\infty m(\{x : f(x) > t\})\, dt.$$

6. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Prove that the image of any set of measure zero has measure zero.

7. Compute

$$\lim_{k\to\infty} \int_0^k (1 - \frac{x}{k})^k e^{x/3}\, dx.$$

Justify your answer with appropriate convergence theorems.

8. Let $E = [0,1]$ and let $f_k, f \in L^1(E)$ for each $k \in \mathbb{N}$, such that

(i) $f_k(x) \to f(x)$ a.e. on E;

(ii) $||f_k||_1 \to ||f||_1$ as $k \to \infty$.

Show that

$$\lim_{k\to\infty} \int_E |f_k(x) - f(x)|\, dm(x) = 0.$$