Joint Program Exam, May 2004

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test m and m_n denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dy or dt, referring to the variable to be integrated.

Part 1.

DO ALL PROBLEMS IN PART ONE.

1. For each of the following, give an example, or explain briefly why no example exists.

(a) A non-measurable subset of the Cantor no-middle-thirds set.

(b) A sequence of functions $f_n \in L^1(\mathbb{R})$ such that f_n converges to zero uniformly on every compact subset of $\mathbb{R}$, but $\int f_n dm = 1$ for all n.

(c) A function $f \in L^1([0,1])$ with f not equal to zero on a set of positive measure, but satisfying $\int_0^x f(t)\,dt = 0$ for almost every $x \in [0,1]$.

2. Are the following statements true or false? Justify!

(a) Let $f \geq 0$ be bounded and measurable on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} f\,dm = \inf \int_{\mathbb{R}} \phi\,dm,$$

where the infimum is taken over all simple measurable functions ϕ with $f \leq \phi$.

(b) If $f \in L^1(\mathbb{R})$ and f is continuous, then $f(x) \to 0$ as $x \to \pm\infty$.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. Let $f : [0,1]^2 \to \mathbb{R}$ be bounded and such that $f(x,\cdot)$ is continuous for all x and $f(\cdot,y)$ is continuous for all y. Show that g defined by

$$g(x) = \int_0^1 \frac{f(x,y)}{y^{1/2}}\,dy$$

is continuous on $[0,1]$.

2. Let Y be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^{10} : x \in Y\}$ also has measure zero.

3. Let $f \geq 0$ on $[0,1]$ be measurable.

(a) Show that $\int_{[0,1]} f^n\,dm$ converges to a limit in $[0,\infty]$ as $n \to \infty$.

(b) If $\int_{[0,1]} f^n\,dm = C < \infty$ for all $n = 1,2,\ldots$, then prove the existence of a measurable subset B of $[0,1]$ such that $f(x) = \chi_B(x)$ for almost every x. Here χ_B denotes the characteristic function of B.

4. Let $1 \leq p < \infty$, $E \subset \mathbb{R}$ measurable with $0 < m(E) < \infty$ and f measurable function. Show that the function g defined by

$$g(p) := \left(\frac{1}{m(E)}\int_E |f|^p dm\right)^{1/p}$$

is non-decreasing on $[1,\infty)$. Here $m(E)$ denotes the Lebesgue measure of E.

5. Let $\varepsilon > 0$ and

$$f(x) = \begin{cases} x^{1+\varepsilon}\sin\frac{1}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Show that f is absolutely continuous on $[0,1]$.

6. Let n be a positive integer. For $x = (x_1, x_1, \ldots, x_n) \in \mathbb{R}^n$ let

$$\|x\|_\infty = \max\{|x_j|, j = 1,\ldots,n\}$$

and let $E = \{x \in \mathbb{R}^n : \|x\|_\infty \geq 1\}$. Define a function $f : E \to \mathbb{R}$ by $f(x) = \|x\|_\infty^{-(n+1)}$. Prove that

$$\int_E f\,dm_n = n2^n.$$

This can be done by using a result which is known as "layer-cake integration" or the "washer method".