Joint Program Exam, May 2005

Real Analysis

Instructions:

You may use up to 3.5 hours to complete this exam.

Work 7 out of the 8 problems.

Justify the steps in your solutions by referring to theorems by name when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

For each problem you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Notations:

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $m(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure. Instead of dm we sometimes write dx or dt, referring to the variable to be integrated.

1. For $f : (0,1) \to \mathbb{R}$, prove or disprove the following statements:

   (a) If f is continuous a.e., then f is measurable.

   (b) If the set $\{x \in (0,1) : f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable.

2. Show that if $A \subset [a,b]$ and $m(A) > 0$, then there are x and y in A such that $|x - y|$ is an irrational number.

3. Let f, $f_n \in L^2([0,1])$, $n \in \mathbb{N}$. Suppose that $f_n \to f$ a.e. on $[0,1]$. Show that $f_n \to f$ in $L^2([0,1])$ if and only if $\|f_n\|_2 \to \|f\|_2$.

4. Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{1+x^2}\,dx.$$

5. Let $f : [-1,1] \to \mathbb{R}$ be absolutely continuous. Show that $f(E)$ is measurable if $E \subset [-1,1]$ is measurable.

6. Let $\alpha > \beta > 0$ and

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x^\beta}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

   Show that f is of bounded variation on $[0,1]$.

7. Let $f \in L^3(\mathbb{R})$ and $g \in L^{3/2}(\mathbb{R})$. If

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$$

   for $x \in \mathbb{R}$, prove that h is continuous on $\mathbb{R}$.

8. Suppose that $f : \mathbb{R} \to [0,\infty)$ is measurable, that $\phi : [0,\infty) \to [0,\infty)$ is monotonically increasing, absolutely continuous on $[0,T]$ for every $T < \infty$, that $\phi(0) = 0$, and that $\phi(f) \in L^1(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} \phi(f(x))\,dx = \int_0^\infty m(\{x \in \mathbb{R} : f(x) > t\})\phi'(t)\,dt.$$