JOINT PROGRAM EXAM, MAY 2006

REAL ANALYSIS

Instructions:

You may use up to $3\frac{1}{2}$ hours to complete this exam.

On each page of your solutions, write down your student ID number, and the number of the problem being answered.

Justify the steps in your solutions by referring to theorems by name when appropriate, and by verifying the hypothesis of these theorems. You do not need to reprove the theorems you used.

For each problem you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit then two "half solutions" to two problems.

Notations:

R denotes the set of real numbers, $m(A)$ refers to the Lebesgue measure of set A, "a.e." means almost everywhere with respect to Lebesgue measure. Instead of dm we sometimes write dx or dt, referring to the variable to be integrated.

PART I.

Do all the problems in Part I.

1. Prove or disprove: if two continuous functions $f(x), g(x) : R \to R$ are equal almost everywhere with respect to Lebesgue measure, then they are equal everywhere.

2. Find the total variation $V_{-1}^{2}(f)$ of the function $f(x) = x(x+1)(x-1)$ on the interval $[-1, 2]$. Justify briefly, but convincingly.

3. Let f_n, $n = 1, 2, ...$, be a sequence of non-negative Lebesgue measurable functions on R such that

$$\int f_n dm \leq \frac{1}{2^n}.$$

   Let $f = \sum_{n=1}^{\infty} f_n$. Show that $f \in L^1(0, \infty)$.

4. Prove or disprove: if $f \in L^1(0,\infty) \cap L^5(0,\infty)$, then $f \in L^3(0,\infty)$.

PART II.

Do 4 out of 5 problems in Part II.

1. Let f be continuous and positive on $[0,1]$. Prove that the region

$$\Omega = \{(x,y) : 0 \le x \le 1, 0 \le y \le f(x)\}$$

is Lebesgue measurable and

$$m(\Omega) = \int_0^1 f(x)dx,$$

$\int_0^1 f(x)dx$ being the Riemann integral.

2. Find

$$\lim_{n\to\infty} \int_0^\infty \frac{1}{1 + x^{\frac{n}{\ln n + 2006}}} dx.$$

Justify your answer.

3. Let $\mu(X) < \infty$. Prove that a non-negative measurable function $f(x)$ belongs to $L^1(X,\mu)$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu \{x \in X : f(x) \ge 2^n\} < \infty.$$

4. Prove or disprove: the function

$$f(x) = \begin{cases} \frac{1}{\ln x} & x > 0, \\ 0 & x = 0, \end{cases}$$

is absolutely continuous on $[0, \frac{1}{2}]$.

5. Let $f \in C^1[a,b]$. Show that

$$V_a^b(f) = \int_a^b |f'(t)|dt,$$

$V_a^b(f)$ being the total variation of f over the interval $[a,b]$, $\int_a^b |f'(t)|dt$ being the Riemann integral.