Joint Program Exam in Real Analysis

May 8, 2018

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally. You need to do 7 of the 8 problems for full credit. Do not submit solutions for more than 7 problems.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure. Instead of $d\mathrm{m}$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\|\cdot\|_p$ denotes the norm on $L^p(X, \mu)$. We also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when I is a subset of $\mathbb{R}$.

1. Suppose $f(x) = x^3$. Show that, if $E \subset [0,1]$ and $\mathrm{m}(E) = 0$, then $\mathrm{m}(f(E)) = 0$.

2. Let f be a bounded, measurable function defined on a finite interval $[a,b]$. Compute
$$\lim_{\alpha \to \infty} \int_a^b f(x) \cos(\alpha x) dx.$$

3. Let $f \in L^1([0,1])$. Evaluate
$$\lim_{n \to \infty} \int_{[0,\,1]} n \ln \left(1 + \frac{|f(x)|^k}{n^k} \right) dx$$
for $k = 1$ and $k = 2$.

4. Suppose $f : [0,1] \to \mathbb{C}$ is continuous and almost everywhere differentiable and that $f' \in L^1([0,1])$. Show that f is absolutely continuous on $[0,1]$, if that is so on $[a,b]$ whenever $0 < a < b \leq 1$.

5. Let $k \mapsto f_k : \mathbb{R}^n \to \mathbb{C}$ be a sequence of integrable functions. Suppose that this sequence converges almost everywhere to the integrable function $f : \mathbb{R}^n \to \mathbb{C}$. Show that $\int_{\mathbb{R}^n} |f_k - f| d\mathrm{m}$ tends to 0 if and only if $\int_{\mathbb{R}^n} |f_k| d\mathrm{m}$ tends to $\int_{\mathbb{R}^n} |f| d\mathrm{m}$.

6. Suppose $f \in L^1(\mathbb{R})$. Show that
$$g(x) = \int_{\mathbb{R}} \mathrm{e}^{-(x-y)^2} f(y) dy$$
is well-defined for any $x \in \mathbb{R}$ and that $g \in L^p(\mathbb{R})$ whenever $1 \leq p \leq \infty$.

7. Show that the set of Lebesgue points of a function $f \in L^1(\mathbb{R}, \mathrm{m})$ contains the points of continuity of f.

8. Suppose X is uncountable. Let $\mathcal{M}$ be the collection of those subsets of X which are either countable or have a countable complement.

   (a) Show that $\mathcal{M}$ is a σ-algebra.

   (b) Let $\mu : \mathcal{M} \to [0, \infty]$ be the counting measure and define
   $$\nu(A) = \begin{cases} 1 & \text{if } A^c \text{ is countable} \\ 0 & \text{if } A \text{ is countable} \end{cases}.$$
   Show that $\nu \ll \mu$ but that no Radon-Nikodym derivative exists. Why is, nevertheless, the Radon-Nikodym theorem not violated?