Joint Program Exam in Real Analysis

May 7, 2019

Instructions:

1. Print your student ID (but not your name) and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 8 problems. All problems are weighted equally.

4. For each problem which you attempt try to give a complete solution and justify carefully your reasoning. Completeness is important: a correct and complete solution to one problem will gain more credit than two half solutions to other problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $\mathrm{m}(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}^d$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure unless noted otherwise. Instead of $d\mathrm{m}$ we sometimes write dx, dt, etc. referring to the variable to be integrated. $L^p(X, \mu)$ denotes the Lebesgue space of order p with respect to the positive measure μ and $\|\cdot\|_p$ denotes the norm on $L^p(X, \mu)$. We also use the abbreviation $L^p(I)$ for $L^p(I, \mathrm{m})$ when I is a subinterval of $\mathbb{R}$.

1. Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sqrt{x}\log(x) & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0. \end{cases}$$

   Show that f is absolutely continuous.

2. Let the positive numbers p, q, and r be related by $1/p + 1/q + 1/r = 1$ and suppose μ is a positive measure on a set X. Let $f \in L^p(X,\mu)$, $g \in L^q(X,\mu)$, and $h \in L^r(X,\mu)$. Prove that $\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r$.

3. Find $\lim_{n\to\infty} \int_0^\infty \frac{n^{1/2}\exp(-nx^2)}{1+x^2} dx$.

4. Given $p > 1$, find a function f such that $f \in L^r([0,1])$ for $r < p$ but $f \notin L^s([0,1])$ for $s \geq p$. Find another function g such that $g \in L^r([0,1])$ for $r \leq p$ but $g \notin L^s([0,1])$ for $s > p$.

5. Show that

$$\int_0^1 \sqrt{x^4 + 5x^2 + 6}\ dx \leq \sqrt{70/9}.$$

6. Let μ be the counting measure on $\mathbb{N}$ and $f_n, f : \mathbb{N} \to \mathbb{R}$. Show that $f_n \to f$ in measure if and only if $f_n \to f$ uniformly. Recall that a sequence f_n converges in measure to f if, for every positive δ, we have $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) = 0$.

7. Define $f : \mathbb{R} \times (0,\infty)$ by $f(x,t) = \exp(-x^2/(2t))/\sqrt{2\pi t}$. It is known that $\int_{\mathbb{R}} f(\cdot,t) = 1$ regardless of $t > 0$ and that $g(x,t) = 2\partial f/\partial t = \partial^2 f/\partial x^2$. Let $s > 0$ and show that

$$\int_{-\infty}^{\infty}\int_s^\infty g(x,t)dtdx \neq \int_s^\infty \int_{-\infty}^{\infty} g(x,t)dxdt.$$

   What can you conclude about the function g?

8. Let μ be the counting measure on $\mathbb{R}$.

   (a) Describe the class of integrable functions.

   (b) Show that the spaces $L^p(\mathbb{R},\mu)$ for $p \in [1,\infty]$ are ordered by inclusion.