Joint Program Exam, September 1988

Real Analysis

INSTRUCTIONS

The test has two parts. Part I involves statements which can either be proved with a short proof or disproved with a counterexample. If you decide to disprove the statement, you are not required to prove that your counterexample is in fact a counterexample,but you must give it very explicitly. You must work all of Part I.

Part II involves statements which you should prove. You should give complete solutions to as many problems as you can from Part II, remembering that the examiners are looking for complete solutions, rather than two seriously incomplete solutions.

Part I

Prove or disprove the following statements.

1.Let μ be a positive measure on a set X. Let $\{f_k\}$ be a sequence of elements of $L^p(X,\mu)$,where $1 \leq p \leq \infty$. Suppose that the sequence converges in $L^p(X,\mu)$ to f. Then $\|f_k\|_p$ converges to $\|f\|_p$.

2.Suppose that f $\in L^2[0,1]$. Then f $\in L^3[0,1]$.

3.Suppose that the sequence $\{f_n\}$ of elements of $L^1[0,1]$ converges pointwise almost everywhere to the function g, and that the sequence $\{f_n\}$ converges in $L^1[0,1]$ to f. Then for almost every x $\in$ [0,1], $f(x) = g(x)$.

4.Suppose that ρ is a positive measure on a σ-algebra A of subsets of a set X. Suppose that $\{S_n\}$ is a sequence of members of A,such that $S_{n+1} \subset S_n$ for each n,and such that $\cap_{n=1}^{\infty} S_n = \Phi$, the empty set, Then

$$\lim_{n\to\infty} \rho(S_n) = 0.$$

Part II

Prove the following statements.

5.Let g be an absolutely continuous montone function on [0,1]. If $E \subset [0,1]$ is a set of Lebesgue measure 0,show that g(E) has Lebesgue measure 0.

6.Let E be a Lebesgue measurable subset of R. Let $f : E \to R$ be nonnegative and Lebesgue integrable over E. Show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_A f(x)dx < \epsilon$ for every measurable subset A of E with Lebesgue measure less than δ.

7.Prove the following theorem from Fubini's theorem. Be sure to state what measures you are using and what the spaces are. State the version of Fubini's theorem you are using,and make sure you have showed that all hypotheses of this theorem hold.

THEOREM. Suppose that $\{a_{ij}\}_{i,j=1}^{\infty}$ is a set of real numbers indexed by the ordered pairs of positive integers. Suppose that $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty}|a_{ij}|) < \infty$. Then for each j, $\sum_{i=1}^{\infty} a_{ij}$ converges,and for each i, $\sum_{j=1}^{\infty} a_{ij}$ converges. Furthermore, $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij})$ and $\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij})$ both converge and are equal.

8.Let ϕ be a continuously differentiable function from R into R, such that $\phi(x) = 0$ for $x \notin (-1,1)$. Let $f \in L^1(R)$. Define $h(x)$ by

$$h(x) = \int_{-\infty}^{\infty} f(y)\phi(x-y)dy.$$

Show that

a)h is continuous;

b)h is continuously differentiable,and

$$h'(x) = \int_{-\infty}^{\infty} f(y)\phi'(x-y)dy.$$

9.Find, proving all assertions,

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}dx.$$

10.Suppose that f is a Lebesgue integrable function on R. Prove that the function

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

is absolutely continuous.

Joint Program Exam, January 1989

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Problems from each part should be attempted. Do as many as you can do. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Part I. For each of the following give an example or give a brief explanation why no example exists. Complete at least 4 problems.

1.An infinite set which has outer measure zero.

2.A set B of real numbers which is not Lebesgue measurable for which the set $\{x \in B : \text{x is irrational}\}$ is Lebesgue measurable.

3.A function which is Lebesgue integrable on [0,1] but is not Riemann integrable on [0,1].

4.A function whose improper Riemann integral over $(0, \infty)$ exists but whose Lebesgue integral over $(0, \infty)$ does not exists.

5.A sequence $\{f_n\}$ of nonnegative measurable functions on [0,1] satisfying $f_{n+1} \leq f_n$ a.e. for n=1,2,... and $\lim f_n = f$ for almost every $x \in [0, 1]$ with f being Lebesgue integrable, but $\lim \int_{[0,1]} f_n \neq \int_{[0,1]} f$. (The statement $\int_{[0,1]} f_n$ should not be taken to mean that $\int_{[0,1]} f_n < \infty$).

Part II. Prove or disprove. Complete at least 3 problems.

1.If a nonnegative Lebesgue measurable function f is Lebesgue integrable on [0,1], then $f \in L^{\infty}[0,1]$.

2.Let f be a nonnegative Lebesgue integrable function on [a,b]. Define $A_n = \{x \in [a,b] : n \leq f(x) \leq n+1\}$ for n=0,1,2,.... Then

$$\sum_{n=0}^{\infty} n\mu(A_n) < \infty$$

where μ denotes Lebesgue measure on [a,b].

3.Let $f(x) = x^2 \sin(\frac{1}{x})$ for $x \in (0,1]$ and $f(0) = 0$. Then $f(x)$ is of bounded variation on [0,1].

4.Let $\{f_n\}$ be a sequence of nonnegative Lebesgue measurable functions on [0,1]. Suppose that
(i) $f_n \to f$ a.e. in [0,1] and
(ii) $\int_{[0,1]} f_n \leq K$ for all n and some constant K.
Then $f \in L^1[0,1]$ and $\|f\|_1 \leq K$.

5.If $f \in L^1[0,1]$, then $\lim_{n\to\infty} \int_{[0,1]} x^n f = 0$.

Part III. Complete at least 3 problems.

1.Let $p > 1$. Suppose that

(i) $f \in L^p[1, \infty)$ and

(ii) $f \geq 0$ a.e. in $[1, \infty)$.

Define $g(x) = \int_1^\infty f(t) \exp^{-tx} dt$. Show that $g \in L^1[1, \infty)$ and

$$\|g\|_1 \leq \|f\|_p / q^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

2.Suppose that f is a nonnegative Lebesgue integrable function in [0,1] and

$$\int_{[0,1]} |f|^n = \alpha$$

for n=1,2,...and some constant α. Show that $f(x) = [f(x)]^2$ a.e. in [0,1].

3.Let $\{f_n\}$ be a sequence of $L^2[0,1]$-functions which converge pointwise for almost every $x \in [0,1]$ to a function $f \in L^2[0,1]$. Show that $f_n \to f$ in $L^2[0,1]$ if and only if $\|f_n\|_2 \to \|f\|_2$.

4.Let μ be Lebesgue measure on $(-\infty, \infty)$. Prove that if E is a measurable subset of real numbers and if $t \in (-\infty, \infty)$ then $E + t$ is measurable and $\mu E = \mu(E + t)$.

Joint Program Exam, May 1989

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

There are two parts to the exam. You should attempt at most 4 problems from Part I and at most 6 problems from Part II.

Part I. For each of the following give an example or give a brief explanation why no example exists. Complete at least 4 problems.

1.A bounded Lebesgue measurable function f on [0,1] which is not Riemann integrable.

2.A sequence $\{f_n\}_1^\infty$ of Lebesgue measurable functions on [0,1] which converge pointwise to 0, but with $\int_{[0,1]} f_n = 1$ for n=1,2,....

3.An $f \in L^3[0,1]$,but $f \notin L^2[0,1]$.

4.An increasing function which has jump discontinuities at every irrational number.

5.A dense open subset of $[0,1] \times [0,1]$ whose complement in [0,1]x[0,1] has positive Lebesgue measure.

Part II. Do 6 of the following nine problems.

1.Let f be an increasing continuous function on [0,2]. For $x \in [0,1]$ and n=1,2,...,let $f_n(x) = n(f(x+\frac{1}{n}) - f(x))$. Show:

$$\text{(i) } f_n \to f' \text{ a.e. on } [0,1];$$
$$\text{(ii) } \int_{[0,1]} f' \leq f(1) - f(0).$$

2. Assume the usual properties of Lebesgue measure,m,on R^1. Show that if $f \in L^1(R^1)$, then for $x \in R^1$

$$\int_{-\infty}^{\infty} f(x+t)dm(t) = \int_{-\infty}^{\infty} f(x)dm(t)$$

3.Suppose that μ is a positive measure on the σ algebra $\mathcal{B}$, which satisfies $\alpha = \inf\{\mu(A) : A \in \mathcal{B}, \mu(A) > 0\} > 0$. Show that

(a) $L^1(\mu) \subset Ł^{\infty}(\mu)$ and

(b) $L^1(\mu) \subset Ł^2(\mu)$.

4.Let $f(x) = \frac{\sin x}{x}$ for $x > 0$.

(i) Show that $f \in L^1[0,b]$ for all $b > 0$.

(ii) Show that $f \notin L^1[0,\infty)$.

5. Let $f \in L^1(R^1)$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} f(nt)dm(t) = 0,$$

where m denotes Lebesgue measure on [0,1].

6.Let $A = \{x \in [0,1]$: there exists infinitely many pairs of positive integers p and q with $|x - \frac{p}{q}| \leq \frac{1}{q^3}\}$. Show that the Lebesgue measure of A is 0.

7.Let $\{E_k\}_1^{\infty}$ be a sequence of bounded Lebesgue measurable sets in R^n such that $E_1 \supset E_2 \supset ... \supset E_k \supset ....$ Set $E = \bigcap_1^{\infty} E_k$. By using the basic properties of measure show that $\lim_{k\to\infty} m(E_k) = m(E)$, where m denotes Lebesgue measure on R^n.

8.Let c be an element of (0,1). Let X be the two element set: $\{0,1\}$. Let $\mathcal{P}(X)$ be the set of all subsets of X. Let μ be a positive measure on the σ-algebra P(X) with the property that $\mu(\phi) = 0, \mu(\{0\}) = c$, and $\mu(X) = 1$. Show that for every f: $X \to R^1$ we have

$\lim_{\alpha\to\infty} \|f\|_\alpha = \|f\|_\infty$, where $\|f\|_\alpha = (\int_X |f|^\alpha d\mu)^{1/\alpha}$.

9.Let $f \geq 0$ be in $L^1[0,1]$. For $x \in [0,1]$ define $F(x) = \int_{[0,1]} \frac{f(y)}{|x-y|^{1/2}} dm(y)$, where m denotes Lebesgue measure on [0,1]. Show that $F \in L^1[0,1]$.

Joint Program Exam, September 1989

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

There are two parts to the exam. The examiner will grade all 5 problems from Part I and 4 problems from Part II.
This exam consists of 4 pages.

Part I. Prove or disprove the following propositions. Do all 5 problems.

1.If a function f(x) is absolutely continuous on a close interval [a,b], then f(x) is of bounded variation.

2.(i) Let m be the Lebesgue measure on $(-\infty, \infty)$. Let M be a set of measure zero, i.e., m(M)=0. Let $\overline{M}$ be the closure of M in $(-\infty, \infty)$. Then $m(\overline{M}) = 0$.

(ii) Let us define counting measure $\mu(\cdot)$, on $(-\infty, \infty)$ by

$$\mu(M) = \begin{cases} \#(M) & \text{if M is a finite set,} \\ \infty & \text{if M is an infinite set,} \end{cases}$$

where M is a subset of $(-\infty, \infty)$ and $\#(M)$ is the number of elements of M in the finite case. Let $\overline{M}$ be the closure of M in $(-\infty, \infty)$. Then $\mu(\overline{M}) = \mu(M)$ for any subset M of $(-\infty, \infty)$.

3.Let $f_n(x)$ $(n = 1, 2, \cdots)$ be a sequence of continuous functions Lebesgue integrable on $[0, \infty)$ which converges uniformly to a $f(x)$ Lesbesgue integrable on $[0, \infty)$. Then

$$\lim_{n\to\infty} \int_0^\infty |f(x) - f_n(x)| dx = 0.$$

4.Let

$$f(x) = \begin{cases} \frac{\cos(x)}{x} & (-1 \leq x < 0, 0 < x \leq 1) \\ 1 & \text{(x=0)}. \end{cases}$$

Then f(x) is (Lebesgue) integrable on [-1,1].

5.Let h(x) be a Lebesgue integrable function on $(-\infty, \infty)$ and set, for $u \geq 0$,

$$\begin{cases} E_u = \{x \in (-\infty, \infty) / |h(x)| \geq u\}, \\ g(u) = u \cdot m(E_u), \end{cases}$$

where m is the Lebesgue measure. Then g(u) is a bounded function on $[0, \infty)$.

Part II. Do 4 of the following 6 problems. Do not hand in more than 4 problems.

1.Let

$$u_n(x) = \frac{1}{n}(\cos x)^{n-1} \sin x \quad (n = 1, 2, \cdots, 0 \leq x \leq \frac{\pi}{2}).$$

(i) Evaluate $\int_0^{\frac{\pi}{2}} u_n(x)dx$

(ii) Prove: $f(x) = \sum_{n=1}^{\infty} u_n(x)$ exists for all $x \in [0, \frac{\pi}{2}]$ and f(x) is integrable on $[0, \frac{\pi}{2}]$.

2.Let g(x) be an integrable function on $(0, \infty)$. Then prove: For any $\delta > 0$, there exists $R > 0$ such that

$$|\int_0^{\infty} g(x)dx| \leq (1 + \delta)| \int_0^R g(x)dx|$$

3.Let a, b be real numbers such that $0 < a < b < \infty$. Does the limit

$$\lim_{n \to \infty} \int_a^b n \sin \frac{x^2}{n} dx$$

exist? Find the limit if it exists. Prove all assertions.

4.Let $p > 1$ and let $f(x) \in L^p((-1,1))$ with respect to Lebesgue measure, i.e.,

$$\int_{-1}^{1} |f(x)|^p dx < \infty$$

(i) Prove that $f(x) \in L^1((-1,1))$.
(ii) Let $I_n = (-\frac{1}{n}, \frac{1}{n})$ $(n = 1, 2, \cdots)$ and let $\gamma = \frac{p-1}{p}$. Then prove

$$\lim_{n\to\infty} n^\gamma \int_{I_n} |f(x)| dx = 0.$$

5.A real-valued function f(x) on [0,1] is said to be (uniformly) Hölder continuous on [0,1] of order α when f(x) satisfies for all x,y∈[0,1]

$$|f(x) - f(y)| \leq c|x - y|^\alpha,$$

where c is a nonnegative constant.
(i) What function can be Hölder continuous function of order α with $\alpha > 1$?
(ii) Let f(x) be a Hölder continuous function of order 1. Then show that f(x) is differentiable at all points $x \in [0,1]$ and that f(x) satisfies the formula

$$f(1) - f(0) = \int_0^1 f'(x) dx$$

where $f'(x)$ is the derivative of f at x.

6.Let f(t) be a function on [0,1] satisfying the following conditions (a)-(d):
(a) f is a C^2-function on [0,1], i.e.,$\frac{df}{dx}$ and $\frac{d^2 f}{dx^2}$ exist and are continuous on [0,1].
(b) $f(t) > 0$ if $t \neq \frac{1}{2}, t \in [0,1]$.
(c) $f(\frac{1}{2}) = 0$.
(d) $\frac{d^2 f}{dx^2}(\frac{1}{2}) > 0$.

Let $R = \{(t,x) : 0 < t < 1 \text{ and } 0 < x < 1\}$. Is the function $u(t,x) = \frac{1}{\sqrt{f(t)+x^2}}$ integrable on R, i.e., is the double integral

$$\iint_R u(t,x) dt dx < \infty \ ?$$

Prove your answer. (Hint: Consider $F(t) = \int_0^1 \frac{1}{\sqrt{f(t)+x^2}} dx$ and use the formula $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$).

Joint Program Exam, May 1990

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, the symbol "$\mu(E)$" refers to Lebesgue measure on the set E. Also the symbol "R" stands for the real numbers.

Part I. For each of the following give an example or give a brief explanation why no example exists.

DO 4 OF THE FOLLOWING 5 PROBLEMS IN PART I.

1. A sequence of functions $\{f_n\}$ in $L^1(R)$ such that, on each compact set K, $\{f_n\}$ converges to 0 uniformly, but $\int_R f_n dm = 1$ for all n.

2.A subset M of R with measure 0, but whose closure $\overline{M}$ has non-zero measure.

3.A non-measurable subset of the Cantor ternary set.

4.A continuous f(x) defined on $[0, \infty)$ such that f is not Lebesgue integrable over $[0, \infty)$ but such that the improper Riemann integral

$$\lim_{R \to \infty} \int_0^R f(x)dx$$

exists.

5.A measure space $\langle X, \mathcal{B}, \mu \rangle$ for which there is no countable family of subsets $\{X_n\}_{n=1}^{\infty}$ such that all of the following hold:

a) $X_n \in \mathcal{B}$ for each n;

b) $X = \cup_{n=1}^{\infty} X_n$;

c) $\mu(X_n) < \infty$ for each n.

Part II: In this part, give a complete solution to each problem. Do six problems in part II.

1.The function f, defined on R, is said to satisfy a *Lipschitz condition* if there is a constant M such that, for all x,y $\in R$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Prove that if f satisfies Lipschitz condition, then there exists monotone functions h and g such that f(x)=h(x)-g(x), for every x.

2.Let $f_n(x) = \sqrt{n}xe^{-nx^3}$,for $n = 1, 2, 3, \cdots$

(i) Find the maximum value assumed by f_n in the interval [0,1].

(ii) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

3.Let m represent Lebesgue measure on R.

(i) Show that $L^p[0,1] \subset Ł^q$ if $p > q$. (You must show that L^p is a subset of L^q and that the inclusion is strict.)

(ii) Show that, if $f \in Ł^\infty$,then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

4.Compute the quantity $\lim_{n\to\infty} [\int_0^1 e^{-x^2/n} dx]$.
Make sure that you verify your manipulations by referring to known theorems.

5.Let f be a non-negative function defined on R and suppose that the function $\frac{f(x)}{1+x^2} \in L^1(R)$. Assume also that, for each positive integer n,

$$\int_{-\infty}^{\infty} \frac{n^2}{n^2+x^2} f(x)dx \leq 1.$$

Show that $f \in L^1(R)$ and that $\|f\|_1 \leq 1$.

6.Let f and g be integrable functions defined on R, and suppose that, for each $t \in R$, $f(t) \geq 0$ and also $g(t) \geq 0$. The convolution of f and g is a new function $h(x) = f * g(x)$, defined as follows:

$$f * g(t) = \int_R f(t-s)g(s)ds.$$

Prove the basic case of Young's Convolution Theorem: namely, prove that $f * g$ is integrable and that $\|f * g\| \leq \|f\|_1 \|g\|_1$.

7.Suppose that the function f is integrable on $[0, \infty)$ and that,for all $r > 0$,

$$\int_0^r f(x)dx = 0.$$

Prove that f(x)=0 for almost every $x > 0$.

8.Let f be a bounded and measurable function defined on the interval [0,1], and let $0 < \alpha < 1$. For each $x \in R$, set

$$h(x) = \int_0^1 f(t)|x-t|^{-\alpha} dt.$$

Show that

(i) h(x) is well-defined for each $x \in R$ (that is, show that the function $f(t)|x-t|^{-\alpha}$ is integrable.

(ii) h(x) is a continuous function on R. [hint: to prove continuity at $x_0 \in [0,1]$, divide the interval of integration into $\{t \in [0,1] : |t-x_0| < \delta\}$ and $\{t \in [0,1] : |t-x_0| \geq \delta\}$.]

Joint Program Exam, October 1990

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, the symbol "$\mu(E)$" refers to Lebesgue measure on the set E. Also the symbol "R" stands for the real numbers.

DO 8 OF THE FOLLOWING 9 PROBLEMS.

1. Let f(x,y) be a continuous function defined on R^2. Let g, h be real-valued measurable functions on [a,b] ($a, b \in R$). Define the function F by

$$F(t) = f(g(t), h(t)), t \in [a, b].$$

Then, prove that F is a measurable function on [a,b].

2. Give an example of a sequence $\{f_n\}$ of real-valued continuous functions on E=[0,1] such that

(i) $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in E$,and

(ii) $\lim_{n\to\infty} \int_E f_n \neq \int_E f$

Justify your answer.

3. Prove or disprove that if f is a nonnegative and integrable function on R, then

$$\lim_{|x|\to\infty} f(x) = 0.$$

4. Let $\{E_j\}$ be a collection of measurable sets such that $\sum_{j=1}^{\infty} \mu(E_j) < \infty$. Let

$$A = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.$$

What can you say about $\mu(A)$? Justify your answer.

5. Let $f \in L^1(E) \cap L^\infty(E)$. Then, show that for $1 < p < \infty$, $f \in L^p(E)$ and

$$\|f\|_p \le \|f\|_\infty^{1/q} \|f\|_1^{1/p}$$

where $\frac{1}{p} + \frac{1}{q} = 1$

6. Let f(x) be a continuous and bounded function on R, and let g(x) be an integrable function on R. Define

$$F(x) = \int_R f(x-y)g(y)dy, \qquad x \in R.$$

(i) Show that F(x) is well-defined for all $x \in R$;

(ii) Show that F is continuous on R.

7. Let $f \in L^1(R)$, $f_n \in L^1(R)$, $f_n(x) \to f(x)$ a.e. and $\|f_n\|_1 \to \|f\|_1$ as $n \to \infty$. Show that for every measurable subset E of R,

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

8. Show that

$$\lim_{n\to\infty} \int_0^n (1 + \frac{x}{n})^n e^{-3x} dx = \frac{1}{2}.$$

(Justify all your assertions).

9. Let $f(x) = x^2 \cos(x^{-1})$ for $x \in (0,1]$ and $f(0) = 0$. Show that $f(x)$ is absolutely continuous on [0,1].

Joint Program Exam, May 1991

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, the symbol "$\mu(E)$" refers to Lebesgue measure on the set E. Also the symbol "R" stands for the real numbers.

DO 9 OF THE FOLLOWING 10 PROBLEMS.

1. Let $f : [a, b] \to R$ be a differentiable function whose derivative is bounded on [a,b]. Prove that

(a) f is absolutely continuous on [a,b];

(b) there exists $c > 0$ such that $\mu(f(I)) \leq c\mu(I)$ for any interval $I \subset [a, b]$.

2. Let Z be a subset of R with Lebesgue measure zero. Show that the set $\{x^3 : x \in Z\}$ also has measure zero.

3. For each problem below give an example with a brief explanation.

(a) A sequence $\{a_n\}$ in ℓ^3,but not in ℓ^2.

(b) A function $f \in L^2[0, 1]$ with $f \notin L^3[0, 1]$.

(c) A sequence $\{a_{ij}\}$ of real numbers such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

4. Let $E \subset R$ be such that the Lebesgue measure $\mu(E) = 0$. Prove that the interior of E is an empty set.

5. Let N be the set of natural numbers and let $\mathcal{P}(N)$ be its power set. Let $\lambda : \mathcal{P}(N) \to [0, \infty]$ be defined by

$$\lambda(E) = \begin{cases} 0 & \text{if E is a finite subset of N} \\ \infty & \text{if E is an infinite subset of N.} \end{cases}$$

Prove or disprove that λ is a measure.

6. Let $f : [0, \frac{\pi}{4}] \to [0, \infty)$ be defined by

$$f(x) = \begin{cases} 0 & \text{if x=0} \\ \frac{\cos x}{x} & \text{otherwise.} \end{cases}$$

Is f Lebesgue integrable on $[0, \frac{\pi}{4}]$? Justify your answer.

7. Let f be a nonnegative Lebsegue integrable function on [0,1]. For $x \in [0, 1]$, define

$$F(x) = \int_{[0,1]} \frac{f(y)}{|x - y|^{1/2}} dy.$$

Show that $F \in L^1[0, 1]$.

8. Let $E = [0, \infty)$ and let $f_n : E \to R$ defined by

$$f_n(x) = \frac{n^{1/4} e^{-x^2 n}}{1 + x^2}.$$

(a) Prove that $f_n \in L^1(E)$;
(b) find $\lim_{n\to\infty} \int_E f_n(x)dx$.

9. Let $E = [0, 2]$ and let $f : E \to R$ be a nonnegative measurable function. Suppose $\lim_{n\to\infty} \int_E f^n(x)d\mu$ exists and is finite. Prove that $\mu(\{x \in E : f(x) > 1\}) = 0$

10. Let $f_n : [0, 1] \to R$ be defined by

$$f_n(x) = \frac{nx^{n-1}}{2 + x} \quad \text{for all } x \in [0, 1] \text{ and } n \in N.$$

Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \tfrac{1}{3}.$$

Joint Program Exam, October 1991

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

DO 8 OF THE FOLLOWING 9 PROBLEMS.

1. Let

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{for } 0 < x \leq 1 \\ 0 & \text{for x=0} \end{cases}$$

Show that (a) f is bounded and continuous on [0,1].
(b) f in not a function of bounded variation on [0,1].

2. Suppose that f is a continuous on [-1,1]. Show that

$$\lim_{n \to \infty} \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t)dt = f(0).$$

3. Evaluate:

$$\sum_{n=0}^{\infty} \int_{0}^{\frac{\pi}{2}} (1 - \sqrt{\sin x})^n \cos x dx.$$

(Justify your work)

4. Construct a sequnece of continuous functions f_n on [0,1] such that $0 \leq f_n \leq 1$,

$$\lim_{n \to \infty} \int_0^1 f_n(x)dx = 0,$$

and yet the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges for no $x \in [0, 1]$.

5. (1) Let f be a step function on a bounded interval [a,b]. Then prove that

$$(*) \qquad \lim_{n \to \infty} \int_a^b f(x) \sin(nx)dx = 0.$$

(2) Let f be a bounded measurable function on [a,b]. Then prove $(*)$
(3) Let f be an integrable function on $(-\infty, \infty)$. Then prove $(*)$.

6. (1) Prove that $f(t) = \frac{t}{1+t^2}$ is a bounded function on $(-\infty, \infty)$.

(2) Let

$$f_n(x) = \frac{nx}{1+n^2x^2} \qquad (x \in [0,1], n = 1, 2, \cdots)$$

Then prove that $f_n(x)$ does not converge uniformly on [0,1].

(3) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ if it exists.

7. Define $f_n, n = 1, 1, \cdots$ on [0,1] by

$$f_n(x) = \begin{cases} n^\alpha & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0 & \text{otherwise,} \end{cases}$$

where α is a constant such that $1 \le \alpha < 2$

(1) Is there any integrable function Φ on [0,1] such that, for all n=1,2,$\cdots$, $0 \le f_n(x) \le \Phi(x)$ for all $x \in [0,1]$?

(2) Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$ if it exists.

8. (1) Prove the Cauchy-Schwarz inequality for $L^2([0,1])$

(*) $|\int_0^1 f(x)g(x)dx| \le \sqrt{\int_0^1 f(x)^2dx}\sqrt{\int_0^1 g(x)^2dx}$ (f,g$\in$L$^2([0,1])$,f,g,are real).

(2) Prove that if the equality holds in $(*)$ there exist $a, b \in R, a^2 + b^2 \ne 0$ such that $af(x) = bg(x)$ a.e. x in [0,1].

(3) Let $f \ge 0$ be Lebesgue measurable on [0,1] and

$$\int_0^1 (f(x))^2dx = \int_0^1 (f(x))^3dx = \int_0^1 (f(x))^4dx < \infty$$

Use (1) and (2) to show that $f = f^2$ a.e.

9. Assume that $f \in L^p(-\infty, \infty)$ for some $1 \le p < \infty$. Define $\alpha(t) = \mu(\{x \in R : |f(x)| > t\})$ for $0 \le t < \infty$ where μ denotes Lebesgue measure. Let

$$E = \{(x,t) \in R^2 : 0 \le t < |f(x)|\}.$$

Then show that

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \int_0^{\infty} pt^{p-1}\alpha(t)dt.$$

[Hint: Compute $\int_{-\infty}^{\infty}\int_0^{\infty} pt^{p-1}\chi_E(x,t)dtdx$, where χ_E is the characteristic function of E.]

Joint Program Exam, May 1992

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

DO any 8 of the 9 problems given. Be sure to indicate which 8 are to be graded.

Notation: Throughout the exam, the symbol "$\mu(E)$" refers to Lebesgue measure on the set E. Also the symbol "R" stands for the real numbers.

1. Let $f : [1, \infty) \to R$ be function defined by $f(x) = \sqrt{x}$. Show that
   (a) f is absolutely continuous on [1,4];
   (b) if $A \subset [1, \infty)$ and $\mu(A) = 0$, then $\mu(f(A)) = 0$.

2. Let A be an open and bounded subset of R^2. Then prove that there exists a sequence $\{K_n\}$ of compact sets such that for each $n \in N$,
   (i) $K_n \subset K_{n+1}$, $K_n \subset A$ and
   (ii) $\lim_{n\to\infty} \mu(K_n) = \mu(A)$.

3. Let $f \in L^1(R)$. Show that

$$\lim_{n\to\infty} \int_R (\sin(x))^n f(x)dx$$

exists and determine its value. Justify all your assertions.

4. Let $f \in L^1(R)$. Prove that the function $F(x) = \int_{-\infty}^{x} f(t)dt$ is continuous.

5. Let $f : [1, \infty) \to R$ be a continuous function such that $f(x + 1) = f(x)$ for all $x \geq 0$. Then show that

$$\lim_{n\to\infty} \int_0^1 2xf(nx)dx = \int_0^1 f(x)dx$$

6. Let $f : R \to R$ be a function defined by

$$f(x) = \begin{cases} \sqrt{n} & \text{if } n \leq x \leq n + 1/n^2, n \in N \\ 0 & \text{otherwise} \end{cases}$$

Show that f is Lebesgue integrable on R but f^2 is not.

7. Show that

$$\lim_{n\to\infty} \int_0^1 \frac{\sqrt{n}x + 2nx^2}{1 + (nx)^3} dx = 0$$

8. If f is of bounded variation on [0,1], show that

$$\int_0^1 |f'| \leq V[0, 1].$$

Show that if equality holds in this inequality, then f is absolutely continuous on [0,1]. (V[0,1] denotes the variation of f over [0,1])

9. Let $f \in L^1(R)$ and let

$$\varphi_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt, h > 0.$$

Prove that φ_h is Lebesgue integrable on R and that $\|\varphi_h\|_1 \leq \|f\|_1$.

Joint Program Exam, May 1993

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, the symbol "$m(E)$" refers to Lebesgue measure on the set E. We use symbols "dm" and "dx" interchangeably. Also the symbol "R" stands for the real numbers.

DO 8 OF THE FOLLOWING 10 PROBLEMS.

1. Suppose f,g are integrable on $(0, \infty)$ and that, for all $r > 0$,

$$\int_0^r f(x)dx = \int_0^r g(x)dx.$$

Prove that f(x)=g(x) for almost all $x > 0$.

2. Let $f \in L^2(R)$ i.e. f is square-integrable. Prove that

$$\lim_{n\to\infty} \int_n^{n+1} f(x)dx = 0.$$

3. Prove or disprove the following statement. Suppose $\{f_n\}$ is a sequence of Lebesgue measurable functions on R such that

(i) $f_n(x)$ converges for all $x \in R$,

(ii) $0 \leq f_n(x) \leq 1$ for all x, and $\int_{-\infty}^{\infty} f_n(x)dx$ is bounded.

Then

(iii) $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dm = \int_{-\infty}^{\infty} \lim\limits_{n\to\infty} f_n(x)dm.$

4. Evaluate

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{e^{-|x|^2/n}}{1+x^2}dx$$

Justify your assertions.

5. Prove that

(a) $\int_0^b \frac{|\sin x|}{x}dx < \infty$ for every $0 < b < \infty$

(b) $\int_0^\infty \frac{|\sin x|}{x}dx = \infty$

6. A function f(x) is Lipschitz on R if there is a number $M < \infty$ such that

$$|f(x) - f(y)| \leq M|x-y|, \text{for all x,y} \in R.$$

Prove that f(x) is Lipschitz if and only if f(x) is absolutely continuous and its derivative is bounded almost everywhere.

7. (a) Find a set $E \subset R$ such that $m(\overline{E}) > m(E)$.
   (b) Find a set $E \subset R$ such that $m(int(\overline{E})) < m(E)$.

Here, $\overline{E}$ is the closure of E and int(F) is the interior of F.

8. Suppose f(x) and g(x) are Lebesgue measurable functions on [0,1]. Prove the following generalization of Hölder's inequality:
if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 \le p, q \le \infty$, then $\|fg\|_r \le \|f\|_p \|g\|_q$.
Here $\|f\|_p = [\int_0^1 |f(x)|^p dx]^{\frac{1}{p}}$.

9. Let r_n be an enumeration of rational numbers in R.
   (a) Show that $R \setminus \cup_{n=1}^{\infty} (r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2})$ is never empty.
   (b) Show that $R \setminus \cup_{n=1}^{\infty} (r_n - \frac{1}{n}, r_n + \frac{1}{n})$ can be empty or non-empty,depending on how the rationales are enumerated.

10. By integration $e^{-xy} \sin x$ with respect to x and y, show that

$$\lim_{n\to\infty} \int_0^n \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(You may assume the results in 5.)

Joint Program Exam, September 1993

Real Analysis

INSTRUCTIONS

You may take up to 3.5 hours to complete eight of ten problems give. Don't turn in more than eight (or else indicate clearly which problems are to be graded). Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, m denotes Lebesgue measure. Sometimes dm(x) or dm(t) indicate also which variable is the integration variable.

1. Suppose that f is a nonnegative, Lebesgue measurable function on [0,1]. Prove that

$$\lim_{n\to\infty}\int_{[0,1]} f^n dm$$

exists (as a finite number) if and only if $m(\{x \in [0,1] : f(x) > 1\}) = 0$.

2. Compute

$$\lim_{n\to\infty}\int_{(1,\infty)} \frac{\log(1+nx)}{1+x^2\log n} dm(x).$$

Justify all your assertions.

3. Prove or disprove the following statement:
Let the functions f_n be in $L^1([0,1], m)$ for $n = 1, 2, \cdots$ Assume that $f_n(x) \to f(x)$ a.e. with respect to m and that

$$\int_{[0,1]} |f_n| dm \to \int_{[0,1]} |f| dm < \infty.$$

Then

$$\int_{[0,1]} |f_n - f| dm \to 0.$$

4. Let μ be a measure on a measurable space X and assume that $1 \leq p < r < q \leq \infty$. Prove that

$$L^p(X,\mu) \cap L^q(X,\mu) \subset L^r(X,\mu).$$

5. Prove that

$$\int_{[0,\infty)} \sqrt{1+\sin x}\exp(-x) dm(x) \leq \sqrt{\frac{3}{2}}.$$

6. Let $f \in L^1([0,\infty), m)$. For $x \geq 0$ define the function F as follows

$$F(x) = \int_{[x,\infty)} f(t)\exp(x-t) dm(t).$$

Prove directly that $F \in L^1([0,\infty), m)$.

7. Let f be a nonnegative, Lebesgue measurable function on R and let $\alpha(y) = m(\{x \in R : f(x) > y\})$ for all $y \geq 0$. Show that for $0 < p < \infty$

$$\int_R f(x)^p dm(x) = p \int_{[0,\infty)} y^{p-1} \alpha(y) dm(y).$$

8. Prove or disprove the following statement:
Let $f_n(x)$ be Lebesgue measurable function on [0,1] such that

$$f_1(x) \geq f_2(x) \geq \cdots \geq 0.$$

Then $\lim_{n\to\infty} f_n(x)$ exists and

$$\lim_{n\to\infty} \int_{[0,1]} f_n dm = \int_{[0,1]} \lim_{n\to\infty} f_n dm$$

9. Prove or disprove the following statement:
If φ is a convex function on (a,b) and f is an integrable function such that $a < f(x) < b$ for all x in a measure space (X, μ) then

$$\varphi(\int_X f d\mu) \leq \int_X (\varphi \circ f) d\mu.$$

10. Let $X = \{0, 1\}$ and $0 < t < 1$. Define a measure μ on X by $\mu(\{0\}) = t$ and $\mu(\{1\}) = 1 - t$. Show that

$$\lim_{p\to 0} \|f\|_p = \exp\,(\int_X \log|f| d\mu)$$

if $f \in L^\infty(X, \mu)$. (Here we used the definitions $\log(0) = -\infty$ and $\exp(-\infty) = 0$.)

Joint Program Exam, May 17, 1994

Real Analysis

INSTRUCTIONS

You have up to three and a half hours. You may attempt all of the nine problems. Solving seven of the nine problems given will give a full score.

Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, m denotes Lebesgue measure and $\|\cdot\|_p$ denotes the norm in L^p with respect to the measure under consideration.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measure space $(\mathcal{X}, \mathcal{B}, \mu)$. Define the set $E = \{x \in \mathcal{X} | \lim f_n(x) \text{ exists } \}$. Show that E is a measurable set.

2. Suppose E is a bounded Lebesgue measurable subset with $m(E) > 0$. Prove that there exist $x_1, x_2 \in E$ such that

   (a) $x_1 - x_2$ is an irrational number.

   (b) $x_1 - x_2$ is a non-zero rational number.

3. Show that

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \frac{(-1)^m n}{n + nm^2 + 1} = \sum_{m=1}^{\infty} \frac{(-1)^m}{1 + m^2}.$$

4. Find

$$\lim_{n\to\infty} \int_0^1 \frac{(nx)^2}{(1+x^2)^n} dx.$$

5. Suppose f is Lebesgue integrable on R. Define

$$h(x) = \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2} dy.$$

(a) Show that h(x) is well defined for each real x and $|h(x)| \leq \|f\|_1$.

(b) Show that h(x) has limits as $x \to \pm\infty$ and compute these limits.

(c) Show that h is differentiable at every point and give a formula for the derivative. What about higher derivatives?

6. Let f be an essentially bounded Lebesgue measurable function defined in [0,1]. Show that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

7. Suppose f is Lebesgue integrable on R. Let

$$\phi_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt, \quad h > 0.$$

(a) Prove that ϕ_h is integrable and $\|\phi_h\|_1 \leq \|f\|_1$.

(b) Show that $\|\phi_h - f\|_1 \to 0$ as $h \downarrow 0$.

8. Suppose f is Lebesgue integrable on R and let

$$g(x) = \int_{-\infty}^{\infty} \frac{f(y)}{(x-y)^2 + 1} dy.$$

Show that $g \in L^p(R)$ for every p, $1 \leq p \leq \infty$, and estimate $\|g\|_p$ in terms of $\|f\|_1$.

9. Suppose $f_j, j = 1, 2, \cdots$ are all Lebesgue integrable on R, that $f_j \to f$ a.e. as $j \to \infty$, and that $\|f_j\|_1 \to A$ as $j \to \infty$.

(a) Show that f is integrable on R and that $\|f - f_j\|_1 \to A - \|f\|_1$.

(b) Must $A = \|f\|_1$? Give proof or counterexample.

Joint Program Exam, September 13, 1994

Real Analysis

INSTRUCTIONS

You have up to three and a half hours. You may attempt all of the nine problems. Complete answer to seven of the the nine problems will give a full score.

Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, m denotes Lebesgue measure. Sometimes dm(x) or dm(t) indicate also which variable is the integration variable.

1. Let $E \subset R$ be a Lebesgue measurable set with a positive measure. Show that for $0 < \alpha < 1$, there exists an open interval I such that $m(E \cap I) > \alpha m(I)$.

2. Prove that

$$\int_1^\infty \frac{\sqrt{1+x^3}}{x^4} dx \leq \sqrt{\frac{7}{10}}.$$

3. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. Suppose that f is an integrable function. Show that $\int |f - f_n| dm \to 0$ if and only if

$$\int |f_n| dm \to \int |f| dm.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with a positive measure μ and $I = [t_1, t_2]$. Suppose that the functions $f(\cdot, t)$ are measureable and that $\frac{\partial f(x,t)}{\partial t}$ exists for every $t \in I$ and almost every $x \in X$ (with respect to μ). Furthermore, assume that there is a $g \in L^1(X, \mu)$ such that $|\frac{\partial f(x,t)}{\partial t}| \leq g(x)$ almost everywhere (with respect to μ). Prove that

$$\frac{d}{dt} \int_A f(\cdot, t) d\mu = \int_A \frac{\partial f(\cdot, t)}{\partial t} d\mu,$$

for all $A \in \mathcal{M}$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with a positive measure μ. Let f be a nonnegative function on X such that for any positive integer n there are two measurable functions g_n and h_n such that $g_n(x) \leq f(x) \leq h_n(x)$, for all $x \in X$, and $\int_X (h_n - g_n) d\mu < \frac{1}{n}$. Prove that f is measurable and that

$$\int_X f d\mu = \lim_{n \to \infty} \int_X g_n d\mu$$

6. Prove that if f and g are absolutely continuous on [a,b], then so is fg, and

$$\int_a^b (fg' + f'g)dx = f(b)g(b) - f(a)g(a).$$

7. Let $f * g(x) = \int_R f(x-y)g(y)dy$, where $f \in L^1(R)$ and $g \in L^1(R)$.
(a) Show that $f * g$ is well defined and in $L^1(R)$.
(b) Show that if f is continuous with compact support, then $f * g$ is continuous.

8. Let

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{x} \neq 0 \\ 0, & \text{x} = 0, \end{cases}$$

and

$$g(x) = \begin{cases} x^2 \sin(1/x^2), & \text{x} \neq 0 \\ 0, & \text{x} = 0, \end{cases}$$

Which of the above functions is of bounded variation on the interval [-1,1]? Justify your assertions.

9. Suppose f and g are two positive measurable functions on [0,1] such that $f(x)g(x) \geq 1$, for all $x \in [0,1]$. Prove that

$$(\int_0^1 f(x)dx)(\int_0^1 g(x)dx) \geq 1.$$

Joint Program Exam, May, 1995

Real Analysis

INSTRUCTIONS

You have up to three and a half hours. You may attempt all of the 10 problems. Full credit will be awarded for answering any 8 of the 10 problems given. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, m denotes Lebesgue measure and $\|\cdot\|_p$ denotes the norm in L^p with respect to the measure under consideration.

1. Let $\{f_n\}_{n=1}^{\infty}$ and f be in $L^1[0,1]$. Suppose that $f_n \to f$ a.e. on [0,1] and $\|f_n\|_1 \to \|f\|_1$. Show that $\|f_n - f\|_1 \to 0$.

2. Let $E = [0,1] \times [0,1]$ and $f(x,y) = \frac{xy}{(x^2+y^2)^2}$ for $0 < x, y \leq 1$ and 0 otherwise. Show that $f \notin L^1(E)$.

3. Suppose that $f_n \to f$ in $L^1[0,1]$ and suppose that $h : R \to R$ is continuous. Prove or disprove: $h \circ f_n \to h \circ f$ in $L^1[0,1]$.

4. Let f be nonnegative a.e. on [0,1] and satisfy $f^n \in L^1[0,1]$ for all $n = 1, 2, \cdots$. Suppose that $Limsup_n \|f^n\|_1 < \infty$.
Prove that $f(x) \leq 1$ for a.e. x in [0,1].

5. Show that

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not absolutely continuous on [0,1].

6. Assume that $\{f_n\}$ and f are measurable on [0,1] and that $f_n(x) \geq 0$ a.e..
Suppose that $f_n \to f$ a.e. on [0,1]. Show that $\int_0^1 f_n(x)e^{-f_n(x)}dx \to \int_0^1 f(x)e^{-f(x)}dx$.

7. A function $f : R \to R$ is called Lipschitz on R if there exists a constant M, such that for all x,y in R, $|f(x) - f(y)| \leq M|x - y|$. Prove that if f is Lipschitz on R and if $E \subset R$ satisfies $m(E) = 0$, then $m(f[E]) = 0$. (Note: $f[E] = \{f(e) : e \in E\}$.)

8. (a) Use Fubini's theorem on $R \times R$ to give a sufficient criterion that for an array of real numbers, $\{a_{ij}\}_{i,j=1}^{\infty}$, we have $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} a_{ij}) = \sum_{j=1}^{\infty}(\sum_{i=1}^{\infty} a_{ij})$.

(b) Give an example of real numbers $\{a_{ij}\}_{i,j=1}^{\infty}$, for which the sums in (a) are not equal.

9. Let N denote the positive integers. Define $E = \{x \in [0,1] : |x - \frac{p}{q}| < \frac{1}{q^3}$, for infinitely many $(p,q) \in N \times N\}$. Show that m(E)=0.

10. Let $\mathcal{M}$ be a subspace of $L^2[0,1]$ with the following property; there is a constant C so that if $f \in \mathcal{M}$, then $|f(x)| \leq C\|f\|_2$ for a.e. $x \in [0,1]$. Let $f_1, f_2, \cdots, f_n$ be an orthonormal set in $\mathcal{M}$.

(a) Show that $\sum_{j=1}^{n} |f_j(x)|^2 \leq C^2$ for a.e. $x \in [0,1]$.

(b) Show that the dimension of $\mathcal{M}$ is bounded above by C^2.

Joint Program Examination
Real Analysis
Fall 1995

Time: Three and One Half Hours

Instructions: Completeness is your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct answer to one problem will gain more credit than solutions to two problem, each of which is only "half-correct".

Full credit will be awarded for answering correctly any eight of the ten problems given.

Notation: $\mathbb{R}$ denotes the set of real numbers, $m(E)$ refers to the Lebesgue measure of the set $E \subset \mathbb{R}$, and "a.e." means almost everywhere with respect Lebesgue measure.

Question 1 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative functions on $[0,1]$ which converge pointwise to f, and suppose that $f_n \leq f$ for each n. Prove that

$$\int f \, dm = \lim_n \int f_n \, dm.$$

Question 2 Let $f \in L^1(\mathbb{R})$, and define

$$g(x) = \int_{-\infty}^{x} f \, dm.$$

Prove that g is continuous.

Question 3 Evaluate:

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} \left(1 - \sqrt{\sin x}\right)^n \cos x \, dx.$$

Question 4 Evaluate:

$$\lim_{n \to \infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)} \, dx.$$

Question 5 Let $\mathbb{N}$ denote the positive integers, and let $\alpha > 2$ be a real number. Define

$$E = \{x \in [0,1] \colon \ |x - p/q| < q^{-\alpha} \text{ for infinitely many pairs } (p,q) \in \mathbb{N}^2\}.$$

Prove that $m(E) = 0$.

Question 6 Let E be a subset of $\mathbb{R}$, and let $0 < \alpha < 1$. Prove:

(a) If $m(E \cap I) \geq \alpha m(I)$ for all open intervals $I \subset \mathbb{R}$, then $m(E) = \infty$.

(b) If $m(E \cap I) \leq \alpha m(I)$ for all open intervals $I \subset \mathbb{R}$, then $m(E) = 0$.

Question 7 Let $f \in L^1([0,1])$ and suppose f is nonnegative. Show that

$$\int_0^1 \frac{f(y)}{|x-y|^{1/2}}\, dy$$

is finite for a.e. $x \in [0,1]$.

Question 8 Suppose that f is a nonnegative and absolutely continuous function on $[0,1]$. Prove or disprove: $\sqrt{f}$ is absolutely continuous.

Question 9 Let f and g be nonnegative functions on $[0,1]$ satisfying

$$fg \geq 1.$$

Prove that

$$\int_0^1 f\, dm \cdot \int_0^1 g\, dm \geq 1.$$

Question 10 Let $f \in L^\infty([0,1])$. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Joint Program Exam, May, 1996

Real Analysis

INSTRUCTIONS

You have up to three and a half hours. You may attempt all of the 10 problems. Full credit will be awarded for answering any 8 of the 10 problems given. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, m denotes Lebesgue measure and $\|\cdot\|_p$ denotes the norm in L^p with respect to the measure under consideration and "measurable" and "integrable" refer to Lebesgue measure when no other measure is indicated.

1. In each of the following prove the statement if it is true, or provide a counter-example if it is false:

(a) Let $E = [0, 1]$; then $\mathcal{L}^{\infty}(E) \subset \mathcal{L}^{1}(E)$.

(b) For every sequence $\{E_n\}$ of real, measurable sets

$$m\big(\cup_{n=1}^{\infty} E_n\big) = \lim_{N\to\infty} m\big(\cup_{n=1}^{N} E_n\big).$$

(c) Let $f : R \to R$ is non-negative and integrable, then f is bounded.

2. Let $f : R \to R$ be measurable. Define $g : R \to R$ by.

$$g(x) = \begin{cases} 0, & \text{if f(x) is rational} \\ 1, & \text{otherwise.} \end{cases}$$

Show that g is measurable.

3. Let μ represent counting measure on the real numbers; that is, if $E \subset R$, then either $\mu(E)$ is the number of elements in E, if E is a finite set of points, or $(E) = \infty$, if E is an infinite set.

(a) Give a simple characterization of what it means for a function $f : R \to R$ to be integrable with respect to μ.

(b) What set inclusions hold among the spaces $\mathcal{L}^{p}(R, \mu)$, with $p \geq 1$?

4. Let $\{f_n\}$ b e a sequence of non-negative measurable functions on [0,1]. Suppose that

(a) $\lim_{n\to\infty} f_n(x)$ a.e. in [0,1], and

(b) for all n and some constant K

$$\int_0^1 f_n dm \leq K.$$

Show that $f \in \mathcal{L}^{1}[0, 1]$ and $\int_0^1 |f| dm \leq K$.

5. For $x \in R$ and $n = 1, 2, ...$ let

$$f_n(x) = (1 - \frac{x}{n})^n.$$

Prove that

$$\lim_{n\to\infty} \int_0^n f_n(x) e^{x/2} dx = 2.$$

(You may assume without proof that $\lim_{n\to\infty} f_n(x) = e^{-x}$.)

6. Let $f : R \to R$ be measurable, and continuous at x. Prove that f is integrable on a neighborhood of x and

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x).$$

7. Suppose that the function $f : [0,1] \to R$ is integrable and that, for all $r \in [0,1], \int_0^r f dm = 0$. Prove that $f = 0$ almost everywhere.

8. Show that if $\alpha \geq 0$, the function $f(x) = x^\alpha$ is absolutely continuous on every bounded subinterval of $[0, \infty)$.

9. Let $f : R \to R$ be integrable. Define a function Let $g : R \to R$ by

$$g(x) = \begin{cases} f(x - \frac{1}{x}), & \text{if x} \neq 0 \\ 0, & \text{if x} = 0. \end{cases}$$

Prove that $g(-1/x) = g(x)$ for $x \neq 0$, and

$$\int_R f dm = \int_R g dm.$$

10. Use the Fubini theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt \qquad (x > 0)$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Your use of the Fubini theorem should be carefully justified.

Joint Program Exam, Fall 1996

Real Analysis

INSTRUCTIONS

You have up to three and a half hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam the symbol "$\mu(E)$" refers to Lebesgue measure of the set E and "R" stands for the real numbers.

1. Let E $= \{x \in [0,1] : x = 0.a_1a_2a_3\cdots, a_n = 2 \text{ or } 3\}$ and $0.a_1a_2a_3\cdots$ denotes the decimal expansion of x. Show that E is measurable and compute its Lebesgue measure.

2. Let E be a subset of [0,1] with Lebesgue measure, $\mu(E) = 0$, and let $g : [0,1] \to R$ be absolutely continuous. Show that $\mu[g(E)] = 0$.

3. Let : $(0,1) \to R$ be defined by

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ is rational in lowest term} \\ 0, & \text{if x is irrational.} \end{cases}$$

Show that:
E=$\{x \in (0,1) :$ h is discontinuous at $x\}$ has Lebesgue measure zero;

4. Let f be a nonnegative integrable function defined on a set E. Show that the Lebesgue measure

$$\mu(\{x \in E : f(x) = +\infty\}) = 0$$

5. Compute

$$\lim_{n\to\infty} \int_{2-\frac{1}{n}}^{2} e^{-t^2} d\mu(t).$$

Be sure you justify all steps.

6. Let f(x) be defined by

$$f(x) = \begin{cases} 3x & \text{if } \ 0 \le x \le \frac{1}{3} \\ 1 & \text{if } \ \frac{1}{3} < x < \frac{2}{3} \\ -3x+3 & \text{if } \ \frac{2}{3} \le x \le 1 \end{cases}$$

Prove that $\lim\limits_{n\to\infty} \int_0^1 [\cos(\pi f(x))]^{2n} d\mu(x)$ exists and find its limit.

7. Let $f : [0,1] \to R$

$$f(x) = \begin{cases} 0, & \text{if x is rational} \\ x, & \text{otherwise.} \end{cases}$$

(i) Prove or disprove that f is of bounded variation on [0,1];

(ii) Is it true that f is in $L^1[0,1]$, but not in $R[0,1]$? Justify your answer (R[0,1] denotes the set of Riemann integrable functions)

8. Let E=[0,1], and let $f_k, f \in L^1(E)$, for each $k \in N$. Suppose

(a) $f_k(x) \to f(x)$ a.e. on E; and

(b) $\|f_k\|_1 \to \|f\|_1$ as $k \to \infty$.

Show that $\lim\limits_{k\to\infty} \int_E |f_k(x) - f(x)| d\mu(x) = 0$.

JOINT PROGRAM EXAM
REAL ANALYSIS
May 1997

Instructions: You may take up to three and a half hours to complete the exam. Completeness in your answers is very important. Justify **each** of your steps by referring to theorems by name if appropriate or by providing a brief statement of the theorem.

An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Full credit can be gained with 8 essentially complete and correct solutions.

Notation: By $L^p(\mu)$ we denote the Lebesgue spaces associated with a measure space (X, μ). Lebesgue measure on $\mathbb{R}$ and its subsets is denoted by m. If X is a subset of $\mathbb{R}$ and $\mu = m$ we use the notation $L^p(X)$ instead of $L^p(\mu)$. We write ℓ^p for the Lebesgue spaces associated with the counting measure on $\mathbb{N}$.

1. Give an example for each of the following objects or a brief explanation why it does not exist.
   (a) A set A of real numbers which is not Lebesgue measurable but for which the set $B = \{x \in A : x \text{ is irrational}\}$ is Lebesgue measurable.
   (b) A function on $\mathbb{R}$ which is Lebesgue integrable but not Riemann integrable.
   (c) A sequence f_n of functions in $L^1([0,1])$ which converges pointwise to zero but satisfies $\int_0^1 f_n dm = 1$.

2. State Fatou's Lemma and the Dominated Convergence Theorem. Prove them using the Monotone Convergence Theorem.

3. Suppose $f \in L^1(\mu)$. Prove: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\int_E f d\mu| < \varepsilon$ whenever $\mu(E) < \delta$.

4. Let f be absolutely continuous in $[\delta, 1]$ for each $\delta > 0$ and continuous and of bounded variation on $[0, 1]$. Prove that f is absolutely continuous on $[0, 1]$.

5. Evaluate
$$\lim_{n\to\infty} \int_0^1 \frac{x^2}{x^2 + (1 - nx)^2} dx.$$

6. Find a function f on $(0, \infty)$ such that $f \in L^p((0, \infty))$ if and only if $1 < p < 2$.

7. Show that the product of two absolutely continuous functions on the interval $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

8. (a) If $g \in L^1(\mathbb{R})$ show that there is a bounded measurable function f on $\mathbb{R}$ such that $\|f\|_\infty > 0$ and
$$\int_{\mathbb{R}} fg dm = \|g\|_1 \|f\|_\infty.$$
(b) If $g \in L^\infty(\mathbb{R})$ show that for each $\varepsilon > 0$ there is an integrable function f on $\mathbb{R}$ such that
$$\int_{\mathbb{R}} fg dm \geq (\|g\|_\infty - \varepsilon)\|f\|_1.$$

9. Suppose $1 \leq p \leq q \leq \infty$. Prove or disprove:
(a) $L^p([0, 1]) \subset L^q([0, 1])$.
(b) $\ell^p \subset \ell^q$.

10. Suppose $f \in L^1(\mathbb{R})$ and define
$$g(x) = \int_{\mathbb{R}} f(y) \exp(-(x - y)^2) dy.$$
Show that $g \in L^p(\mathbb{R})$ for every $p \in [1, \infty]$ and estimate $\|g\|_p$ in terms of $\|f\|_1$. (Note that $\int_{\mathbb{R}} \exp(-x^2) dx = \sqrt{\pi}$.)

JOINT PROGRAM EXAM
REAL ANALYSIS
FALL 1997

Instructions: Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems each of which is "half correct".

Notation: Throughout the exam, $\mathbb{R}$ stands for the set of real numbers.

1. For a measurable subset E of $\mathbb{R}^n$, prove or disprove:
   (a) If E has Lebesgue measure 0, then its closure has Lebesgue measure 0.
   (b) If the closure of E has Lebesgue measure 0, then E has Lebesgue measure 0.
2. (i) Prove that convergence in L^1 implies convergence in measure. (Give a precise statement, then prove it.)
   (ii) Is the converse true?
3. Denote the ternary Cantor set in $[0,1]$ by C. Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 0, & x \in C \\ x^2, & x \notin C. \end{cases}$$
   Prove that f is a Lebesgue measurable function.
4. Show that the hyperelliptic integral
$$\int_2^\infty \frac{xdx}{\sqrt{(x^2-\varepsilon^2)(x^2-1)(x-2)}}$$
   converges to the elliptic integral
$$\int_2^\infty \frac{dx}{\sqrt{(x^2-1)(x-2)}}$$
   when ε tends to zero.

5. Show that
$$\left(\int_0^1 \frac{x^{1/2}dx}{(1-x)^{1/3}}\right)^3 \leq \frac{8}{5}.$$
6. Evaluate
$$\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-\mathrm{e}^{-t})^n \mathrm{e}^{-t^2} dt.$$
7. Either give an example of the specified mathematical object or quote a theorem that proves that no such object can exist.
(i) A sequence f_n of functions converging to 0 uniformly on $[0,1]$, but such that $\int_0^1 f_n dm \geq 1$ for all n (here m denotes the Lebesgue measure).
(ii) A sequence f_n of functions converging to 0 uniformly on $\mathbb{R}$, but such that $\int_{\mathbb{R}} f_n dm \geq 1$ for all n (here m denotes the Lebesgue measure).
(iii) A sequence f_n of positive functions converging pointwise to a function f such that $\liminf \int_0^1 f_n < \int_0^1 f$.
8. Let Σ denote the σ-algebra of subsets of $\mathbb{R}$ generated by the set $\{0\}$ (i.e. Σ is the smallest σ-algebra in $2^{\mathbb{R}}$, which contains $\{0\}$).
(i) Prove that every $E \in \Sigma$ is Lebesgue measurable.
Let μ denote the restriction of the Lebesgue measure m (on $\mathbb{R}$) to Σ [i.e. $\forall E \in \Sigma : \mu(E) = m(E)$]. It is easy to show (you do not have to!) that μ is a measure on Σ.
(ii) Determine the dimension of $L^2(\mathbb{R}, \mu)$.
Help: Σ is so 'small' that one can list its elements.

Joint Program Examination in Real Analysis

May 1998

Instructions: You may use up to $3\frac{1}{2}$ hours to complete this exam. For each problem which you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than "half correct" solutions to two problems.

Justify the steps in your solution by referring to theorems by name, when appropriate. (If you cannot remember the name of a theorem but do remember the statement, you may give the statement instead). When you refer to named theorems (e.g., Fubini's Theorem or the Monotone Convergence Theorem) in a problem, you do not need to reprove the named theorem.

Throughout this examination, the symbol $\mathbb{R}$ denotes the real numbers. When dx is used in integration, the measure will be Legesgue measure on the real numbers.

Part I.

For each problem below, a correct solution consists either of an example, as called for by the problem, or an explanation why no such example exists. The explanation should generally refer to a known theorem.

DO 4 OF THE PROBLEMS IN PART ONE.

1. A sequence, f_n, of functions in $L^1[0,1]$ such that $|f_n(x)| \le 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$.

2. A sequence, f_n, of functions in $L^1[0,\infty]$ such that $|f_n(x)| \le 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = 1$.

3. A subset S of $\mathbb{R}$ with Lebesgue measure 0 whose closure $\overline{S}$ has non-zero measure.

4. A bounded Lebesgue measurable function f on $[0,1]$ which is <u>not</u> Riemann integrable.

5. A sequence (x_n) which is in $\ell^{1+\epsilon}$, for all $\epsilon > 0$, but which is not in ℓ^1.

6. An absolutely continuous monotonic function defined on $[0,1]$ such that $f(0) = 0$, $f(1) = 1$, and $f'(t) = 0$ a.e. on $[0,1]$.

Part II.

Try to give a complete solution for each of the problems which you attempt in this part.

DO 5 OF THE PROBLEMS IN PART TWO.

1. Let $1 \le p < q < \infty$. Show that $L^q[0,1]$ is a proper subset of $L^p[0,1]$. (Be sure to show that the containment is strict.)

2. A function f defined on $\mathbb{R}$ is said to satisfy a *Lipschitz condition* if there is a constant M such that, for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le M|x - y|.$$

   Prove that if f satisfies a Lipschitz condition, then there exist monotone functions h and g such that $f(x) = h(x) - g(x)$, for all x.

3. Let $A \subset [0,1]$ be a non-measurable set. Let $B = \{(x,0) \in \mathbb{R}^2 \mid x \in A\}$.

   (a) Is B a Lebesgue measurable subset of $\mathbb{R}^2$? Prove your answer.

   (b) Can B be a closed subset of $\mathbb{R}^2$ for some such A? Prove your answer.

4. Evaluate the following limit and justify your calculations.

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)}\, dx.$$

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose that $f \colon X \to \mathbb{R}$ is measurable and satisfies $\|f\|_\infty < \infty$. Show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

6. Prove or disprove each of the following:

   (a) Let $-\infty < a < b < \infty$ and let f and g be absolutely continuous on $[a,b]$. Then fg is absolutely continuous on $[a,b]$.

   (b) Let f and g be absolutely continous on $\mathbb{R}$. Then fg is absolutely continuous on $\mathbb{R}$.

7. Let f and g be integrable functions defined on $\mathbb{R}$. Assume that $f(t) \geq 0$ and $g(t) \geq 0$, for all $t \in \mathbb{R}$. The *convolution* of f and g is defined as follows:

$$f * g(t) = \int_{\mathbb{R}} f(t-x) g(x)\, dx.$$

   Prove that $f * g$ is integrable and that $\|f * g\|_1 = \|f\|_1 \, \|g\|_1$

8. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = 2(x-y)e^{-(x-y)^2} u(x), \quad \text{where } u(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

   You may use without calculation the following two facts: $\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dy \right] dx = 0$ and $\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dx \right] dy = \sqrt{\pi}$.

   Determine the value of the integral: $\int_{\mathbf{R}^2} |f(x,y)|\, dx\, dy$. Justify your answer.

9. Show that

(a) $x^{-1}\sin x \notin L^1[0,\infty]$, that is

$$\int_0^\infty \frac{|\sin x|}{x}\,dx = \infty.$$

(b)

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}\,dx \quad \text{exists and is finite.}$$

JOINT PROGRAM EXAM
REAL ANALYSIS
FALL 1998

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is half correct.

Notation: Throughout the exam the symbol "$\mu(E)$" refers to Lebesgue measure of the set E. Also the symbol "$\mathbb{R}$" stands for the real numbers. When dx is used in integration, the measure will be the Lebesgue measure on the real numbers.

DO THE FOLLOWING 8 PROBLEMS

Note: Make sure that you carefully justify your claims.

1. Let E be a measurable subset of $\mathbb{R}$. Suppose there exists a number $c \in (0,1)$ such that

$$\mu(E \cap I) \leq c\mu(I)$$

for all intervals I in $\mathbb{R}$. Show that $\mu(E) = 0$.

2. Evaluate the following limit and justify each of your steps.

$$\lim_{n\to\infty} \int_1^\infty \frac{\ln(1+nx)}{1+x^2 \ln n}\, dx.$$

3. Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. Define

$$h(x) = \int_{\mathbb{R}} f(x-t)g(t)dt.$$

Show that

$$\|h\|_1 \leq \|f\|_1 \|g\|_\infty.$$

4. Let $f, \{f_k\} \in L^2$. Suppose (i) $f_k \to f$ a.e. and (ii) $\|f_k\|_2 \to \|f\|_2$. Show that

$$\|f - f_k\|_2 \to 0 \quad \text{as} \quad k \to \infty.$$

5. Let $f \in L^1(0,\infty)$. Suppose $f > 0$ a.e. Define $F(t) = t^{-1}\int_0^t f(x)dx$. Prove that $F \notin L^1(0,\infty)$.

6. Prove or disprove whether there exists a sequence $\{f_n\}$ in $L^1[0,1]$ such that
(i) $|f_n(x)| \leq 2$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$;
(ii) $f_n(x) \to 0$ a.e. and
(iii) $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1/2$.

7. If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that

$$f = 0 \text{ a.e. in } E.$$

8. Let $f(t,x)$ be defined on $[0,1]\times\mathbb{R}$ such that $f(t,\cdot) \in L^1(\mathbb{R})$. Suppose there exists $g \in L^1(\mathbb{R})$ such that

$$|f(t,x)| \leq g(x) \quad \text{for all} \quad (t,x) \in [0,1] \times \mathbb{R}.$$

If $t_0 \in [0,1]$ and, for almost every $x \in \mathbb{R}$, $f(t,x)$ is continuous at t_0, then show that

$$F(t) = \int_{\mathbb{R}} f(t,x)dx$$

is also continuous at t_0.

JOINT PROGRAM EXAM

REAL ANALYSIS

MAY, 1999

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems each of which is "half correct".

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Notation: Throughout the exam the symbol $m(E)$ refers to Lebesgue measure of the set E and $\mathbf{R}$ stands for the real numbers. The notation $\int_{[0,1]} f(x)dx$ is used for the Lebesgue integral of $f(x)$, while the Riemann integral is denoted by $\int_0^1 f(x)dx$.

1. Prove that if $m^*(E) = 0$, $E \subset \mathbf{R}^n$, then E is Lebesgue measurable ($m^*(E)$ is the outer measure of E).

2. Let $f : [0,1] \to [0,1]$ and $f \in C^1[0,1]$. Use the definitions of Lebesgue measure and Riemann integral to show that the Lebesgue measure of the domain $A = \{(x,y),\ 0 \leq x \leq 1,\ 0 \leq y \leq f(x)\}$ in $\mathbf{R}^2$ is given by the Riemann integral of $f(x)$, i.e.,

$$m(A) = \int_0^1 f(x)dx. \tag{1}$$

3. Let f be a fixed non-negative Lebesgue integrable function on $\mathbf{R}^n$. For any Lebesgue measurable set $E \subseteq \mathbf{R}^n$, define $\mu(E) = \int_E f dx$ (integral with respect to the Lebesgue measure on $\mathbf{R}^n$).

(i) Prove that μ is a measure on the σ-algebra $\mathcal{M}$ of all Lebesgue measurable subsets on $\mathbf{R}^n$.

(ii) Give an example of a measure on $\mathcal{M}$, which can not be obtained by the construction given above. Justify.

4.

(i) Let $f : \mathbf{R}^k \to \mathbf{R}^m$ be Lebesgue measurable, and $g : \mathbf{R}^m \to \mathbf{R}^l$ be continuous. Prove that $g \circ f : \mathbf{R}^k \to \mathbf{R}^l$ is measurable.

(ii) Prove that any monotonic function on (a,b) is Lebesgue measurable.

5. Let $f \in C^1[0,1]$ and positive. Prove that the Riemann and Lebesgue integrals coincide, i.e.,

$$\int_0^1 f(x)dx = \int_{[0,1]} f(x)dx. \tag{2}$$

6. Find the limit

$$\lim_{n\to 0} \int_{[0,1]} \cos(x^n)dx.$$

Be sure you justify all steps.

7. Let $f \in L^1(\mathbf{R}, m)$ and $E_1 \subseteq E_2 \subseteq ...$ be measurable subsets of R. Prove that

$$\lim_{k\to\infty} \int_{E_k} f dx$$

exists, $\int_{E_k} f dx$ being the Lebesgue integrals.

8. Each of the problems below describes a mathematical object with certain properties. If the object exists, give an example. If it does not, give a theorem and/or a short explanation that proves that it does not exist:

(i) An absolutely continuous function f defined on [0,1] and a sequence of subsets E_n of $[0,1]$ such that

$$\frac{m(f(E_n))}{m(E_n)} > n.$$

(ii) A sequence of measurable functions $f_n : [0,1] \to [0,\infty]$ such that

$$\int_{[0,1]} \liminf f_n dx > \liminf \int_{[0,1]} f_n dx.$$

(iii) A sequence of measurable functions $f_n : [0,1] \to [0,\infty]$ such that

$$\int_{[0,1]} \liminf f_n dx < \liminf \int_{[0,1]} f_n dx.$$

9.

(i) Prove that any function of bounded variation can be represented as a difference of two nondecreasing functions.

(ii) Prove or disprove: Any function of bounded variation can be represented as a difference of two *strictly* increasing functions.

10.

(i) Prove that

$$\int_0^{\pi/2} \sqrt{xsinx}dx \leq \frac{\pi}{2\sqrt{2}} \tag{3}$$

(Hint: Hölder's inequality).

(ii)Prove that in fact we have the *strict* inequality in (3).

Joint Program Exam in Real Analysis
September 14, 1999

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. Completeness in your answers is very important. Justify your steps by referring to theorems by name when appropriate. An essentially complete and correct solution to one problem will gain more credit than solutions to two problems each of which is "half correct".

Notation: Throughout the exam the symbol $\mu(E)$ refers to Lebesgue measure of the set E, and for simplicity, $d\mu(x)$ will be denoted by dx. Also the symbols "$\mathbb{Q}$" and "$\mathbb{R}$" stand for the rational and real numbers respectively.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } f(x) \in \mathbb{Q} \\ x & \text{otherwise.} \end{cases}$$

(a) Show that the set $E = \{x \in \mathbb{R} : f(x) \in \mathbb{Q}\}$ is measurable.

(b) Show that h is measurable.

2. Let f be nonnegative and integrable on $\mathbb{R}$. Define $\phi : (0, \infty) \to \mathbb{R}$ by

$$\phi(t) = \mu(\{x : f(x) \geq t\}).$$

Show that

(i) if $0 < a < b < \infty$, then ϕ is of **bounded variation** on $[a, b]$.

(ii) $\lim_{t\to\infty} t\phi(t) = 0$.

(iii) Is ϕ necessarily **absolutely continuous** on $[a, b]$? (why?)

3. Let $E = [0, \infty)$. Prove that $\lim_{n\to\infty} \int_E \frac{x}{1+x^n} dx$ exists, and find its value. Justify all your assertions.

4. Let $E = [0, \infty)$. If $f \geq 0$ on E and $f \in L^1(E)$ then prove or disprove that

$$\lim_{x\to\infty} f(x) = 0.$$

5. Let $E = [1, \infty)$ and suppose $f \in L^2(E)$ such that $f \geq 0$ a.e. in E. Define

$$g(x) = \int_E f(t)\mathrm{e}^{-tx} dt.$$

Show that $g \in L^1(E)$ and

$$2\|g\|_1 \leq \|f\|_2.$$

6. Suppose

(i) $f_n, f \in L^1[0,1]$;

(ii) $f_n \to f$ a.e. in $[0,1]$;

(iii) $\|f_n\|_1 \to \|f\|_1$.

Show that $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

7. Let $f \in L^1[0,1]$ and $f \geq 0$. Show that $\int_0^1 \frac{f(y)}{|x-y|^{2/3}} dy$ is finite for a.e. $x \in [0,1]$.

8. Let $f \in L^1(\mathbb{R}) \cap L^3(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

Joint Program Exam of May 2000 in Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test *a.e.* and dx refer to Lebesgue measure on $\mathbb{R}$.

Part 1.

DO 5 PROBLEMS IN PART ONE. MARK THE ONES TO BE GRADED.

For each of the following statements decide whether it is true or false. Give a succinct proof of your assertion. Sometimes an example may be sufficient.

1) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^{-x} & x \in \mathbb{Q} \\ e^{x} & x \notin \mathbb{Q} \end{cases}$$

is measurable.

2) If the boundary of $\Omega \subset \mathbb{R}^k$ has outer measure zero, then Ω is measurable.

3) The union of two non-measurable subsets of $\mathbb{R}$ is never measurable.

4) If $f_n \in L^1(\mathbb{R})$, $f_n(x) \leq e^{-|x|}$ and $f_n(x) \to 0$ for almost every x then $\limsup \int f_n \, dx \leq 1$.

5) For every finite positive measure μ on $\mathbb{R}$ there exists a non-negative measurable function f such that for all measurable sets E

$$\mu(E) = \int_E f \, dx.$$

6) There exist two sequences $(a_n) \in l^1$ and $(b_n) \in l^2$ such that $(a_n + b_n)$ is neither in l^1 nor in l^2.

Part 2.

DO 5 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1) Suppose $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and prove that:
(a) f is of bounded variation.
(b) $|f|$ is absolutely continuous and

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)| \qquad a.e.$$

2) It is easy to guess the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx.$$

Prove that your guess is correct. Be sure to justify all steps.

3) Find a function $f \in L^1(\mathbb{R})$ which does not belong to any $L^p(\mathbb{R})$ with $p > 1$.

4) Suppose $f(x)$ and $xf(x)$ belong to $L^1(\mathbb{R})$. Prove that

$$\hat{f}(k) = \int e^{ikx} f(x)\, dx$$

is differentiable and that

$$\frac{d}{dk}\hat{f}(k) = \int e^{ikx} ixf(x)\, dx.$$

5) Let $(a_n)_{n\geq 0}$ be a sequence of non-negative real numbers and for each $t \geq 0$ let $N(t) = \#\{n : a_n > t\}$. That is, $N(t)$ is the number of integers n for which $a_n > t$. Show that

$$\sum_{n\geq 0} a_n = \int_0^\infty N(t) dt.$$

Hint: If a_n is considered as the area of a suitable rectangle in $\mathbb{R}^2$ then the left side becomes an integral over $\mathbb{R}^2$.

6) While preparing for his class, Prof. N. read the following definition of the Lebesgue integral $\int_E f d\mu$ of a bounded measurable function f over a measurable set E of finite measure.

"Let f_k be any sequence of measurable simple functions on E that converges uniformly to f (a function being simple if and only if it achieves only finitely many values). Then $\int_E f d\mu \stackrel{def}{=} \lim_{k\to\infty} \int_E f_k d\mu$ (for simple functions the integral has already been defined)."

Being shaky on the concept of uniform convergence, Prof. N. decided to simplify the definition by dropping the word "uniform" from it. Which bounded measurable functions have an integral according to Prof. N.'s definition? Prove your assertion.

Joint Program Exam, May 2001

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test m_n denotes Lebesgue measure on $\mathbb{R}^n$ and 'measurable' is short for 'Lebesgue-measurable'. Instead of dm_1 we write dx.

Part 1.

DO ALL PROBLEMS IN PART ONE.

1. (a) Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable? Justify.
(b) Does there exist a non-measurable subset of $\mathbb{R}$ whose complement in $\mathbb{R}$ has outer measure zero? Justify.
(c) Do there exist two non-measurable sets whose union is measurable? Justify.

2. (a) Let $p > q \geq 1$. Show by example that $L^p([0,1]) \neq L^q([0,1])$.
(b) Show by example that there exist two functions $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ such that $f + g$ is neither in $L^1(\mathbb{R})$ nor in $L^2(\mathbb{R})$.

3. Let f and f_n, $n = 1, 2, \ldots$, be non-negative measurable functions on $[0,1]$ such that f_n converges pointwise to f. Under each of the following additional assumptions, either prove that $\int_0^1 f_n \, dm \to \int_0^1 f \, dm$ or show that this is not generally true. Integrals and convergence are to be understood in the sense of extended real numbers.
(a) $f_n \geq f$ and $f_n \in L^1([0,1])$ for all n,
(b) $f_n \geq f_{n+1}$ for all n,
(c) $f_n \leq f$ for all n.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. (a) Let $a_1, \ldots, a_n$ be positive numbers. Prove that their harmonic mean is bounded by their arithmetic mean, i.e.

$$\left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{a_k} \right)^{-1} \leq \frac{1}{n} \sum_{k=1}^{n} a_k .$$

(b) Characterize the vectors $(a_1, \ldots, a_k)$ for which equality holds in (a).

2. Compute

$$\lim_{n \to \infty} \int_0^1 \frac{n x^{n-1}}{2 + x} \, dx.$$

Make sure that you justify your answer with appropriate convergence theorems.

3. Let $E \in \mathbb{R}^n$ be measurable with $m_n(E) > 0$. Show that $L^2(E) \not\subset L^\infty(E)$.

4. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous with bounded support, i.e. $\{x : f(x) \neq 0\}$ is bounded, and let $g \in L^1(\mathbb{R})$. Define the convolution of f and g by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) \, dy \qquad (x \in \mathbb{R}).$$

(a) Show that $f * g$ is continuous.
(b) If, in addition, f is continuously differentiable, then prove that $f * g$ is continuously differentiable and $(f * g)'(x) = (f' * g)(x)$.

5. Suppose $f \in L^1(\mathbb{R}^2)$ is real-valued. Show that there exists a measurable set $E \subset \mathbb{R}^2$ such that

$$\int_E f dm_2 = \int_{\mathbb{R}^2 \setminus E} f dm_2 .$$

6. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and let

$$E = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

Show that $m_2(E) = 0$.

Joint Program Exam, September 2001

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

For each problem which you attempt try to give a complete solution. A correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Throughout this test m denotes Lebesgue measure on $\mathbb{R}$ and 'measurable' is short for 'Lebesgue-measurable'. Instead of dm we write dx.

1. (a) Suppose that $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ and $A \times B$ is measurable in $\mathbb{R}^2$. Does this imply that A and B are measurable in $\mathbb{R}$?

(b) Does there exist a non-measurable $A \subset [0,1]$ such that $B := \{(x,0) \in \mathbb{R} : x \in A\}$ is a closed subset of $\mathbb{R}^2$?

2. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if x is rational} \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is measurable.
(ii) Prove or disprove that f is of bounded variation on [0, 1].
(iii) Is f Lebesgue integrable, but not Riemann integrable? Justify your answer.

3. Let $E \subset \mathbb{R}$ be measurable and $c \in (0,1)$.

(a) Suppose that $m(E \cap I) \leq cm(I)$ for all intervals I in $\mathbb{R}$. Show that $m(E) = 0$.

(b) What can be said about E if $m(E \cap I) \geq cm(I)$ for all intervals I?

4. (a) Let $f : [a,b] \to \mathbb{C}$ and $g : [a,b] \to \mathbb{C}$ be absolutely continuous. Show that fg is absolutely continuous and that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{for a.e. } x \in [a,b].$$

(b) Let g be absolutely continuous on $[0,\pi]$. Use part (a) to show that

$$\lim_{k\to\infty} \int_0^{\pi} \sin(kx)g(x)\,dx = 0.$$

5. Let $f : \mathbb{R} \to [0,\infty)$ be measurable. Prove that

(a) $E := \{(x,t) \in \mathbb{R} \times [0,\infty) : f(x) > t\}$ is measurable in $\mathbb{R} \times [0,\infty)$,

(b)

$$\int_{\mathbb{R}} f(x)\,dx = \int_0^{\infty} m(\{x : f(x) > t\})\,dt.$$

6. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous. Prove that the image of any set of measure zero has measure zero.

7. Compute

$$\lim_{k\to\infty} \int_0^k (1 - \frac{x}{k})^k e^{x/3}\, dx.$$

Justify your answer with appropriate convergence theorems.

8. Let $E = [0,1]$ and let $f_k, f \in L^1(E)$ for each $k \in \mathbb{N}$, such that

(i) $f_k(x) \to f(x)$ a.e. on E;

(ii) $||f_k||_1 \to ||f||_1$ as $k \to \infty$.

Show that

$$\lim_{k\to\infty} \int_E |f_k(x) - f(x)|\, dm(x) = 0.$$

Joint Program Exam in Real Analysis

September 10, 2002

Instructions: You may take up to $3\frac{1}{2}$ hours to complete the exam. *Do seven problems out of eight*. Completeness in your answers is very important. Justify your steps by referring to theorems by name, when appropriate, or by providing a brief theorem statement. An essentially complete and correct solution to one problem will gain more credit, than solutions to two problems, each of which is "half correct".

Notation: Throughout the exam, "R" stands for the set of real numbers. Notation such as $\int_{[1,0]} f$, $\int_{[1,0]} f(x)\,dx$, etc. is used for Lebesgue integral, while Riemann integral is denoted $\int_0^1 f(x)\,dx$, $\int_0^\infty f(x)\,dx$, etc.

1. Let f: [0, 1] $\to$ R be monotone increasing, that is $f(y) \geq f(x)$ whenever $y > x$. Show that f has at most countably many discontinuities.

2. Let f, f_k be integrable on [0, 1], $k = 1, 2, ?$. Suppose that $f_k \to f$ a.e. and $\int_{[0,1]} |f_k| \to \int_{[0,1]} |f|$. Prove that $\int_{[0,1]} |f_k - f| \to 0$.

3. Find the limit (justify steps): $\lim_{n\to\infty} \int_0^1 \frac{(nx)^2}{(1+x^2)^n}\, dx$.

4. Let f: [0, 1] $\to$ R be Lebesgue−measurable and non−negative, and let m denote one−dimensional Lebesgue measure.

   (a) Show that

$$\int_{[0,1]} f\, dm = \int_0^\infty m(\{x \in [0,1] : f(x) > t\})\, dt.$$

   (b) Suppose in addition that there exists a finite constant C such that

$$m(\{x \in [0,1] : f(x) > t\}) \leq \frac{C}{t}$$

   for all $t > 0$. Show that $f^s \in L^1([0, 1])$ for all $s \in (0,1)$.

5. Let $f(x,y) \geq 0$ be measurable on $\mathrm{R}^n \times \mathrm{R}^n$. Suppose that, for a.e. $x \in \mathrm{R}^n$, $f(x,y)$ is finite for a.e. y. Prove that, for a.e. $y \in \mathrm{R}^n$, $f(x,y)$ is finite for a.e. x.

6. Denote $I = [0, 1]$. Let $f: I \times I \to \mathrm{R}$ be measurable and such that $\int_I \left[\int_I f(x,y)\, dy\right] dx = 1$ and $\int_I \left[\int_I f(x,y)\, dx\right] dy = -1$. Find the range of values of $\iint_{I\times I} |f(x,y)|\, dxdy$ over all such functions f.

7. Show that

$$\left(\int_0^1 \frac{x^{1/2}\, dx}{(1-x)^{1/3}}\right)^3 \leq \frac{8}{5}.$$

8. Let $g \in L^1(\mathrm{R})$ and $G(x) = \int_{\mathrm{R}} g(y)\, e^{-(x-y)^2}\, dy$. Prove that, for any $p \in [1,\infty)$,

$G \in L^p(\mathbb{R})$ and estimate $\|G\|_p$ in terms of $\|g\|_1$.

Joint Program Exam of May, 2003

in Real Analysis

Instructions:

You may take up to three and a half hours to complete this exam.

Work 7 out of the 9 problems. Full credit can be gained with 7 essentially complete and correct solutions.

Justify each of your steps by referring to theorems by name where appropriate, or by providing a brief theorem statement. You do not need to reprove the theorems you use.

For each problem you attempt, try to give a complete solution. A correct and complete solution to one problem will gain more credit than solutions to two problems, each of which is "half-correct".

Notation:

$\mathbb{R}$ denotes the set of real numbers, $m(E)$ refers to the Lebesgue measure of the set $E \subset \mathbb{R}$, "measurable" refers to Lebesgue measure and "a.e." means almost everywhere with respect to Lebesgue measure.

Problem 1.
Give an example or prove non-existence of such.
(a) A subset of $\mathbb{R}$ of measure zero, whose closure has positive measure.
(b) A sequence (f_n) of functions in $L^1[0,1]$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n \, dm \to \infty$.

Problem 2.
(a) Let E be a measurable subset of $\mathbb{R}^2$. Suppose that, for a.e. $x \in \mathbb{R}$, the set $E_x \stackrel{def}{=} \{y \in \mathbb{R} : (x,y) \in E\}$ has measure zero in $\mathbb{R}$. Prove that, for a.e. $y \in \mathbb{R}$, the set $E^y \stackrel{def}{=} \{x \in \mathbb{R} : (x,y) \in E\}$ has measure zero in $\mathbb{R}$.
(b) Let A be a non-measurable subset of $\mathbb{R}^2$ whose intersection with the y-axis is not empty. Can the set $A_0 \stackrel{def}{=} \{y \in \mathbb{R} : (0,y) \in A\}$ be measurable for some such A?

Problem 3.
Let $f \in L^1(\mathbb{R}) \cap L^{17}(\mathbb{R})$. Prove that $f \in L^5(\mathbb{R})$.

Problem 4.
Let $E = [0,\infty)$. Prove that $\lim_{n\to\infty} \int_E \frac{x}{1+x^n} \, dx$ exists, and find its value. Justify all your assertions.

Problem 5.
Let E be a measurable subset of $\mathbb{R}$, and let $f, f_k \in L^1(E)$, $k \in \mathbb{N}$. Suppose that $f_k \to f$ a.e. on E and $\|f_k\|_1 \to \|f\|_1$. Prove that then $f_k \to f$ in $L^1(E)$.

Problem 6.
Let $f \in L^1[0,1]$. Prove that, for a.e. $x \in [0,1]$, $\int_{[0,1]} \frac{f(y)}{\sqrt{|x-y|}} \, dm(y)$ exists and is finite.

Problem 7.
Let f be continuous and strictly increasing on $[0,1]$. Suppose that $m\big(f(E)\big) = 0$ for every set $E \subset [0,1]$ with $m(E) = 0$. Show that f is absolutely continuous.

Problem 8.
Let f be integrable on $[0,1]$. Prove that there exists $c \in [0,1]$ such that $\int_{[0,c]} f\,dm = \int_{[c,1]} f\,dm$.

Problem 9.
Let f be a Lebesgue measurable function on $\mathbb{R}$. Show that:

$$\int_{\mathbb{R}} |f|^3\,dm = 3\int_0^\infty t^2 m\big(\{|f| > t\}\big)\,dt.$$

Joint Program Exam, May 2004

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this test m and m_n denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dy or dt, referring to the variable to be integrated.

Part 1.

DO ALL PROBLEMS IN PART ONE.

1. For each of the following, give an example, or explain briefly why no example exists.

(a) A non-measurable subset of the Cantor no-middle-thirds set.

(b) A sequence of functions $f_n \in L^1(\mathbb{R})$ such that f_n converges to zero uniformly on every compact subset of $\mathbb{R}$, but $\int f_n dm = 1$ for all n.

(c) A function $f \in L^1([0,1])$ with f not equal to zero on a set of positive measure, but satisfying $\int_0^x f(t)\, dt = 0$ for almost every $x \in [0,1]$.

2. Are the following statements true or false? Justify!

(a) Let $f \geq 0$ be bounded and measurable on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} f\, dm = \inf \int_{\mathbb{R}} \phi\, dm,$$

where the infimum is taken over all simple measurable functions ϕ with $f \leq \phi$.

(b) If $f \in L^1(\mathbb{R})$ and f is continuous, then $f(x) \to 0$ as $x \to \pm\infty$.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. Let $f : [0,1]^2 \to \mathbb{R}$ be bounded and such that $f(x,\cdot)$ is continuous for all x and $f(\cdot,y)$ is continuous for all y. Show that g defined by

$$g(x) = \int_0^1 \frac{f(x,y)}{y^{1/2}}\, dy$$

is continuous on $[0,1]$.

2. Let Y be a subset of $\mathbb{R}$ with measure zero. Show that the set $\{x^{10} : x \in Y\}$ also has measure zero.

3. Let $f \geq 0$ on $[0,1]$ be measurable.

(a) Show that $\int_{[0,1]} f^n\, dm$ converges to a limit in $[0,\infty]$ as $n \to \infty$.

(b) If $\int_{[0,1]} f^n\, dm = C < \infty$ for all $n = 1, 2, \ldots$, then prove the existence of a measurable subset B of $[0,1]$ such that $f(x) = \chi_B(x)$ for almost every x. Here χ_B denotes the characteristic function of B.

4. Let $1 \leq p < \infty$, $E \subset \mathbb{R}$ measurable with $0 < m(E) < \infty$ and f measurable function. Show that the function g defined by

$$g(p) := \left(\frac{1}{m(E)} \int_E |f|^p dm\right)^{1/p}$$

is non-decreasing on $[1,\infty)$. Here $m(E)$ denotes the Lebesgue measure of E.

5. Let $\varepsilon > 0$ and

$$f(x) = \begin{cases} x^{1+\varepsilon} \sin \frac{1}{x}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

Show that f is absolutely continuous on $[0,1]$.

6. Let n be a positive integer. For $x = (x_1, x_1, \ldots, x_n) \in \mathbb{R}^n$ let

$$\|x\|_\infty = \max\{|x_j|, j = 1, \ldots, n\}$$

and let $E = \{x \in \mathbb{R}^n : \|x\|_\infty \geq 1\}$. Define a function $f : E \to \mathbb{R}$ by $f(x) = \|x\|_\infty^{-(n+1)}$. Prove that

$$\int_E f\, dm_n = n2^n.$$

This can be done by using a result which is known as "layer-cake integration" or the "washer method".

Joint Program Exam, September 2004

Real Analysis

Instructions. You may use up to 3.5 hours to complete this exam. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

Throughout this exam m denotes Lebesgue measure on $\mathbb{R}$. 'Measurable' is short for 'Lebesgue-measurable'. Instead of dm we sometimes write dx or dy, referring to the variable to be integrated. $L^p(a, b)$ is the L^p space with respect to m on the interval (a, b).

Part 1 below accounts for 40 percent of the exam grade, Part 2 for 60 percent. Separately the questions of Parts 1 and 2 carry equal weight.

Part 1.

DO ALL PROBLEMS IN PART ONE.

Are the following statements true or false? Justify!

1. There is a sequence of measurable subsets E_n of $\mathbb{R}$ with $E_n \subset E_{n+1}$ for $n = 1, 2, \ldots$, such that $m(\cup_n E_n) \neq \lim_{n\to\infty} m(E_n)$.

2. There is a sequence of measurable subsets D_n of $\mathbb{R}$ with $D_n \supset D_{n+1}$ for $n = 1, 2, \ldots$, such that $m(\cap_n D_n) \neq \lim_{n\to\infty} m(D_n)$.

3. There are measurable functions f_n, $n = 1, 2, \ldots$, and f on $[0, 1]$ such that $f_n(x) \to f(x)$ for every $x \in [0, 1]$, but $\int_{[0,1]} f_n \, dm \not\to \int_{[0,1]} f \, dm$.

4. There is a subset A of $\mathbb{R}$ which is not Lebesgue measurable, but such that $B = \{x \in A : x \text{ is irrational}\}$ is Lebesgue measurable.

5. There exists an absolutely continuous function f on $[0, 1]$ such that $f(0) = 0$, $f(1) = 1$, and $f'(t) = 0$ for almost every $t \in [0, 1]$.

Part 2.

DO 4 PROBLEMS IN PART TWO. MARK THE ONES TO BE GRADED.

1. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \left\{ \begin{array}{ll} \sqrt{x} & \text{if } x \text{ is irrational} \\ 0 & \text{otherwise.} \end{array} \right\}$$

   (i) Show that f is measurable.

   (ii) Is f Lebesgue integrable? If yes, find its Lebesgue integral.

   (iii) Is f Riemann integrable? If yes, find its Riemann integral.

2. Suppose that f_n, g_n, f, $g \in L^1(\mathbb{R})$, $f_n \to f$, $g_n \to g$ almost everywhere in $\mathbb{R}$, and $\int_{\mathbb{R}} g_n \, dm \to \int_{\mathbb{R}} g \, dm$ as $n \to \infty$. If $|f_n| \leq g_n$ for all n, prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \, dm = \int_{\mathbb{R}} f \, dm.$$

3. Prove that

$$\int_0^1 \sqrt{x^4 + 4x^2 + 3} \, dx \leq \frac{2}{3}\sqrt{10}.$$

4. Prove or disprove: There is a function f on $(0,1)$ such that $f \in L^p(0,1)$ for all $p \in [1, \infty)$, but $f \notin L^\infty(0,1)$.

5. Let $f \in L^1(0,1)$, $f \geq 0$. Show that

   (i) $\displaystyle\int_0^1 \frac{f(y)}{|x-y|^{1/2}} \, dy < \infty$ for almost every $x \in [0,1]$,

   (ii) $\displaystyle\int_0^1 \frac{f(y)^{1/2}}{|x-y|^{1/4}} \, dy < \infty$ for every $x \in [0,1]$.

6. (i) Let f and g be absolutely continuous on $[0,1]$. Show that fg is absolutely continuous and that

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \quad \text{for almost every } x \in [0,1].$$

   (ii) Let g be absolutely continuous on $[0,1]$. Show that there is a finite constant C (only depending on g) such that

$$\left| \int_0^1 \sin(kx) g(x) \, dx \right| \leq \frac{C}{|k|}$$

   for all non-zero k.

Joint Program Exam, May 2005

Real Analysis

Instructions:

You may use up to 3.5 hours to complete this exam.

Work 7 out of the 8 problems.

Justify the steps in your solutions by referring to theorems by name when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

For each problem you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

Notations:

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, $m(A)$ refers to the Lebesgue measure of the set $A \subset \mathbb{R}$, "measurable" refers to Lebesgue measurable, and "a.e." means almost everywhere with respect to Lebesgue measure. Instead of dm we sometimes write dx or dt, referring to the variable to be integrated.

1. For $f : (0,1) \to \mathbb{R}$, prove or disprove the following statements:

   (a) If f is continuous a.e., then f is measurable.

   (b) If the set $\{x \in (0,1) : f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable.

2. Show that if $A \subset [a,b]$ and $m(A) > 0$, then there are x and y in A such that $|x - y|$ is an irrational number.

3. Let f, $f_n \in L^2([0,1])$, $n \in \mathbb{N}$. Suppose that $f_n \to f$ a.e. on $[0,1]$. Show that $f_n \to f$ in $L^2([0,1])$ if and only if $\|f_n\|_2 \to \|f\|_2$.

4. Find the limit and justify your answer:

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{1+x^2}\, dx.$$

5. Let $f : [-1,1] \to \mathbb{R}$ be absolutely continuous. Show that $f(E)$ is measurable if $E \subset [-1,1]$ is measurable.

6. Let $\alpha > \beta > 0$ and

$$f(x) = \begin{cases} x^\alpha \cos \frac{\pi}{x^\beta}, & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

   Show that f is of bounded variation on $[0,1]$.

7. Let $f \in L^3(\mathbb{R})$ and $g \in L^{3/2}(\mathbb{R})$. If

$$h(x) = \int_{-\infty}^{\infty} f(x-t) g(t)\, dt$$

   for $x \in \mathbb{R}$, prove that h is continuous on $\mathbb{R}$.

8. Suppose that $f : \mathbb{R} \to [0,\infty)$ is measurable, that $\phi : [0,\infty) \to [0,\infty)$ is monotonically increasing, absolutely continuous on $[0,T]$ for every $T < \infty$, that $\phi(0) = 0$, and that $\phi(f) \in L^1(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} \phi(f(x))\, dx = \int_0^\infty m(\{x \in \mathbb{R} : f(x) > t\}) \phi'(t)\, dt.$$