EGR 265, Math Tools for Engineering Problem Solving
April 23, 2014, 10:45am to 1:15pm

Name (Print last name first): ..........................................

Student ID Number: .............. ..............

Final Exam

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Problem 1 (8 points)

Find an explicit solution of the initial value problem

\[ 2yy' = x, \quad y(2) = 1. \]
Problem 2 (10 points)

A liquid is heated to 180°F. It cools down according to Newton’s law of cooling in a surrounding medium of temperature 80°F. The rate of cooling is \( k = -1/3 \).

(a) State the differential equation which governs the temperature of the medium at time \( t \) according to Newton’s law of cooling.

(b) Solve this differential equation with the correct initial value (this can be done either as a separable or a linear equation).

(c) At what time has the temperature dropped to 100°F? (Logarithms do not need to be evaluated.)
Problem 3 (14 points)

Consider the second order differential equation

\[ y'' + 6y' + 9y = 3x - 1. \]  
(1)

(a) Find the general solution of the homogeneous equation corresponding to (1).

(b) Find a particular solution of the inhomogeneous equation (1).
(c) Solve the initial value problem given by (1) and initial conditions $y(0) = 0, \ y'(0) = 0$. 
Problem 4 (10 points + 4 points bonus)

A mass of 4 kg stretches an undamped spring by 10 cm. For simplicity, assume that $g = 10 \text{ m/s}^2$.

(a) Find the spring constant $k$, including its correct unit. Also find the angular frequency $\omega$ of the spring-mass system.

(b) Set up the second order differential equation which governs the motion of the spring-mass system, choosing the $x$-axis to be oriented downwards. Find the general solution of this equation.

(c) Find the particular solution of the equation if the mass is released from 1 meter above the equilibrium position at a downward velocity of 50 cm/s.
(d) (Bonus) A damping force proportional to $\beta$ times the instantaneous velocity is added to the above spring mass system. How does $\beta$ have to be chosen to achieve critical damping?
Problem 5 (10 points)

(a) Find the gradient of \( f(x, y) = \ln(x^2 + y) \).

(b) Evaluate the directional derivative of \( f(x, y) \) at the point with coordinates \((1, 1)\) in the direction of the vector from the point \((1, 3)\) to \((3, 6)\).

(c) Find a unit vector in the direction of steepest decrease of \( f(x, y) \) at the point \((1, 1)\).
Problem 6 (8 points + 4 points bonus)

(a) Determine the equation of the tangent plane to the graph of $x^2 + 3y^2 + 2z^2 = 9$ through the point $(2, 1, 1)$.

(b) (Bonus) Are there points in 3D space at which the tangent plane to $x^2 + 3y^2 + 2z^2 = 9$ is horizontal (i.e. parallel to the $xy$-plane)? If yes, provide all three coordinates for each one of these points.
Problem 7 (8 points)

Find the line integral

\[ \int_C x^2 y \, ds, \]

where \( C \) is the straight line segment connecting the points \((0, 1)\) and \((1, 0)\).
Problem 8 (12 points)

(a) Verify that the force field $\mathbf{F}(x, y) = (y^2 - 2xy)i + (2xy - x^2 + 1)j$ is conservative.

(b) Find a potential function $\phi(x, y)$ for $\mathbf{F}(x, y)$.

(c) Find the work done by the force field $\mathbf{F}(x, y)$ along the curve parameterized by $x = 2/t$, $y = t^2$, $1 \leq t \leq 2$. 
Problem 9 (12 points)

(a) Find the double integral \( \int \int_{R} x^2 \, dA \), where \( R \) is the region in the \( xy \)-plane between the \( x \)-axis and the graph of \( y = 1 - x^2 \).

(b) What is the physical meaning of this integral?
Problem 10 (8 points)

An inhomogeneous lamina of mass density \( \rho(x, y) = x^2 + y^2 \) fills the washer shaped region between the two disks of radius 1 and 2, both centered at the origin. Find the mass of the lamina.