SECTION 3

DYNAMIC SYSTEMS & CONTROL
FORCE CONTROLLED INTERACTION TASKS AND INTUITIVE TEACHING OF A PNEUMATIC MANIPULATOR

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ABSTRACT
A method is presented for the force controlled interaction tasks and intuitive teaching of a pneumatic actuator without using a load cell for force feedback. A sliding mode force controller is designed to provide high bandwidth actuator force tracking using a four-way proportional valve. A unified velocity (free space) and contact force (constrained environment) controller is designed. A novel sigmoid velocity trajectory is planned for velocity command in free space. The contact force is proportional to the velocity immediately before the contact, which lies in the final flat zone of the sigmoid velocity trajectory. Based on this methodology, an efficient and intuitive robot programming method is proposed by directly guiding the robot in the most natural way for force-controlled interaction tasks such as polishing and painting of small lot sizes with a very high number of part variants. The preliminary experimental results are promising.

INTRODUCTION
Pneumatic actuators are natural impedances with true mechanical compliance. Forces are controlled by manipulating the difference of pressures in the two chambers of the actuator, and compliance is provided by the compressibility of air. The natural compliance can be controlled to offer a pneumatic actuator the ability to follow a kinematically constrained trajectory while following a desired interaction force profile. Pneumatic actuators have the additional advantage of being capable of measuring force using pressure sensors instead of a load cell located at the end effector of the actuator.

Today, industrial automation with robots is efficient and fast only for large lot size applications in free space. The use of robots to automate some tasks involving contact with a constrained environment and complex motion planning has not been widespread because they are difficult to program. For force-controlled contact tasks like polishing [1] [2] and applications of small lot sizes with a very high number of part variants like painting [3], efficient and intuitive robot programming has a pressing need. Intuitive programming by directly guide the robot in the most natural way provides the most feasible solution for this need. This idea was originally proposed by Heindl and Hirzinger [4] in 1983. A device was invented for intuitive teaching and programming of motions and forces/torques for a robot. The latest and most successful version of such a device is called SPACE MOUSE [5], which can provide translational and rotational movement increments in 6 DOF. It can also be magnetically fixed on the robot wrist to naturally guide the robot. High price makes it not feasible to integrate it into any industrial robot's wrist since cheap and

NOMENCLATURE

- $V_{a,b}$: volume of chamber a and b
- $m_{a,b}$: mass flow rate of chamber a and b
- $A_v$: valve orifice area
- $\Psi_{a,b}$: area normalized mass flow rate
- $P_{a,d}$: upstream and downstream pressure
- $P_s$: supply pressure

- $A_{a,b}$: cross section area of chamber a and b
- $P_{a,b}$: pressure in chamber a and b
- $P_{atm}$: atmospheric pressure
- $A_r$: cross section area of cylinder rod
- $F_a$: actuator force
- $F_d$: desired actuator force
- $M$: total moving mass
- $B$: viscous friction coefficient
- $x$: rod displacement
- $x_d$: desired displacement
- $F_f$: Coulomb friction force
- $F_e$: external force
- $R$: ideal gas constant
- $T$: air temperature
is the area normalized mass flow rate can be expressed as:

\[ \frac{\dot{m}_{a,b}}{V_{a,b}} = \frac{P_{(a,b)} V_{a,b}}{RT} - \frac{\dot{P}_{(a,b)} V_{a,b}}{V_{a,b}} \]  

The nonlinear relationship between the valve orifice area and mass flow rate is given by:

\[ \dot{m}_a = A_\psi_a(P_a, P_d) \quad \text{for } A_v \geq 0 \]
\[ \dot{m}_b = -A_\psi_b(P_a, P_d) \quad \text{for } A_v < 0 \]

A common mass flow rate model is given by [13]:

\[ \Psi(P_a, P_d) = \begin{cases} \frac{C_i C_f P_a}{\sqrt{T}} & \text{if } \frac{P_a}{P_d} \leq C_v \quad \text{(choked)} \\ \frac{C_i C_f P_a}{\sqrt{T}} \left( 1 - \frac{P_a}{P_d} \right)^{1/k} & \text{otherwise (unchoked)} \end{cases} \]

where \( C_f \) is the discharge coefficient of the valve, \( k \) is the ratio of specific heats, \( C_v \) is the pressure ratio that divides the flow regimes into choked and unchoked flow and \( C_i \) and \( C_2 \) are constants defined as:

\[ C_1 = \sqrt{\frac{k}{R(k+1)}} \left( \frac{2}{k+1} \right)^{(k+1)/(k-1)} \quad \text{and} \quad C_2 = \frac{2k}{R(k-1)} \]

The objective of the force control is to make the actuator force to track a desired force trajectory \( F_d \). The actuator force tracking error is defined as \( e = F_a - F_d \). Taking the derivative of \( F_a \) yields \( \dot{F}_a = \dot{P}_a A_a - \dot{P}_b A_h - \dot{P}_b A_h \). Since \( \dot{F}_a \) is directly related to \( \dot{P}_{(a,b)} \), \( \dot{F}_a \) is directly related to \( \dot{m}_{(a,b)} \), and \( \dot{m}_{(a,b)} \) is directly related to control input \( (A_v = u) \), the dynamic model of the pneumatic actuator is a first order nonlinear system ( \( n = 1 \)) if we do not consider the dynamics of the valve spool position control. This also shows us a very important feature of pneumatic actuators: they are natural impedance. The force output differentiation is dependent on the history of mass flow, which is the flow in the sense of a bond graph. This natural impedance makes a pneumatic actuator a good match to admittance environment.
A control diagram of the pneumatic actuator is shown in Figure 2. This single input dynamic system can be put into standard form as [14]:

\[
\dot{\mathbf{F}} = f(\mathbf{X}) + b(\mathbf{X})u
\]  

(9)

Where \( \mathbf{X} \) is the state vector. The standard time varying surface is defined as \( s = \left( \frac{d}{dt} + \lambda \right)^n e \), for \( n = 1 \), it becomes,

\[
s = e = F_a - F_d = P_a A_a - P_b A_b - P_{atm} A_r - F_d
\]

Taking the derivative of \( s \) gives the sliding mode equation,

\[
\dot{s} = \left( \frac{RT}{V_a} \dot{m}_a - \frac{P_b V_a}{V_a} \right) A_a - \left( \frac{RT}{V_b} \dot{m}_b - \frac{P_b V_b}{V_b} \right) A_b - \dot{F}_d
\]  

(10)

The equivalent control can be solved by setting \( \dot{s} = 0 \):

\[
\frac{RT}{V_a} \dot{m}_a - \frac{P_b V_a}{V_a} A_a - \frac{RT}{V_b} \dot{m}_b - \frac{P_b V_b}{V_b} A_b - \dot{F}_d = 0
\]  

(11)

Substituting Eq. (4) and (5) into Eq. (11) gives,

\[
\frac{A_a}{V_a} \psi_a(P_a, P_d) + \frac{A_b}{V_b} \psi_b(P_a, P_d) RT - \frac{P_b V_a}{V_a} A_a + \frac{P_b V_b}{V_b} A_b - \dot{F}_d = 0
\]  

(12)

Solving Eq. (12) for the equivalence control law,

\[
u_{eq} = \frac{(P_a A_a^2 V_b + P_b A_b^2 V_a) \dot{x} + V_a V_b \dot{F}_d}{RT[A_a V_b \psi_a(P_a, P_d) + A_b V_a \psi_b(P_a, P_d)]}
\]

(13)

Then a discontinuous term is added across the sliding surface to obtain the robust control law [14]:

\[ u = u_{eq} - \kappa \cdot \text{sat}(\frac{s}{\phi}) \]

(14)

The discontinuous term compensates for the model inaccuracies, where \( \kappa \) is the controller gain constant, \( \phi \) is the boundary layer thickness constant and \( \text{sat} \) is a saturation function.

The experimental actuator force tracking performance shown in Figure 3, is up to 30 Hz, which shows that the valve and controller can provide high bandwidth force tracking. In these experiments, the cylinder is fixed around middle stroke position.
FORCE CONTROLLED INTERACTION TASKS WITHOUT FORCE FEEDBACK

Imagine the procedure when one’s hand approaches an object, the fingers are soft and faster speed is used when the hand is further from the estimated contact point; as the hand gets closer, the speed would slow down and minimize the contact force to minimize the likelihood that the contact point would be disturbed or that the hand will be injured. The pneumatic system has the intrinsic compliance like a soft hand, which is something we do not have to control. This biological response will be mimicked through the unique feature of a pneumatic actuator.

The idea is to use a sigmoid velocity function as input to generate the desired actuator force, plus friction compensation and gravity compensation. The desired contact force can be controlled proportionally by the final velocity immediately before contact occurs. Because of the flatness of the sigmoid function when the end effector gets close to the estimated contact position, the contact force is almost independent of the position uncertainty and contact surface stiffness. Let the distance between current position and the estimated contact position be $|x - x_d|$, which is the input of the sigmoid function, and the output of the sigmoid function is the desired velocity. So the desired velocity can be represented as,

$$
\dot{x}_d = \frac{C_1}{1 + e^{-\frac{|x-x_d|}{C_2/C_1}}} + C_4
$$

For example, if $C_1 = 150$, $C_2 = 35$, $C_3 = 7$ and $C_4 = 50$, the sigmoid function is plotted as in Figure 4. By adjusting these four parameters, the desired velocity trajectory can be easily generated. The most important parameter is $C_4$, which determines the final velocity before contact. Therefore, it also determines the final contact force.

$$
F_d = M\ddot{x} + B\dot{x} + F_f - b(\dot{x} - \dot{x}_d)
$$

Eq (16) is similar to a standard impedance control relationship. The position error portion is not included because our input is going to be a velocity trajectory. For contact tasks, generally position tracking is not important along the force control degree-of-freedom. More important thing is that the actuator is going to behave like a passive and dissipative air damper, impact energy is going to be transformed and released quickly, which eliminates big overshoot and bouncing. This corresponds to Hogan’s description about a system imposing passive and dissipative behavior [10] [11], which is not essential to impedance control, but it does offers significant benefits in terms of stability robustness and insensitivity to kinematic errors.

It can be seen that when the contact occurs, both $\dot{x}$ and $\ddot{x}$ become zero. The desired force becomes $F_d = F_f + b\dot{x}_d$, since the actuator force tracking has very high bandwidth, we assume $F_a = F_d$, Substituting this into the system dynamic equation $M\ddot{x} = F_a - B\dot{x} - F_f - F_e$ gives $F_e = b\dot{x}_d$, so by controlling the final flat region velocity of the actuator, the contact force can be closely controlled without knowing the stiffness and the accurate position of the contact surface.

EXPERIMENTAL RESULTS

Experimental results of the force control interaction tasks were carried out using the vertical piston of a pneumatic manipulator that is based on a Festo two degree-of-freedom pick and place pneumatic system. It is shown in Figure 5. The vertical degree-of-freedom (Z direction) is provided by a double acting pneumatic cylinder (SLT-16-100-P-A), which has a stroke length of 100 mm, inner diameter of 16mm and piston rod diameter of 6 mm. A linear potentiometer (Midori LP-
100F) with 100 mm maximum travel is used to measure the linear position of the vertical cylinder. The velocity was obtained from position by utilizing an analog differentiating filter with a 20 dB roll-off at 33 Hz. The acceleration signal was obtained from the velocity signal with a digital differentiating filter with a 20 dB roll-off at 30 Hz. One four-way proportional valve (Festo MPYE-5-M5-010-B) is attached to the two chambers of the vertical cylinder. Two pressure transducers (Festo SDE-16-10V/20mA) are attached to each cylinder chamber, respectively. Control is provided by a Pentium 4 computer with one A/D board (National Instruments PCI-6031E) for analog input channels for sensors and another A/D board (Measurement Computing PCIM-DDA06/16) for analog output channels to control the four proportional valves. A load cell (Transducer Techniques MLP-25) is mounted between the end of the vertical cylinder and the cylindrical peg to monitor the contact force, but is not used for any feedback control purposes.

Figure 5. Experimental Setup of the Pneumatic Manipulator

Detailed experimental results of a case with 90 N desired contact force are shown in Figure 6. It is aluminum (load cell side) to steel (environment side) hard surface contact. Constant Coulomb friction compensation was applied while the viscous friction parameter has been changed from a constant to a function of final contact velocity to accommodate various final velocities and maintain acceptable velocity tracking performance for all cases.

Figure 6 (a-c) Contact task with 90N Desired Contact Force
It can be seen from the velocity tracking in Figure 6 that the velocity tracking is not very accurate sometimes, but the pneumatic actuator can well accommodate the velocity error, and velocity error immediately before the contact shows very little impact on the open loop contact force.

More contact tasks are carried out for nine sample desired contact forces from 10 N to 90 N using the same simple coulomb friction compensation. The results are shown below in Figure 7. The worst case is for 10 N desired force, with a force error of about 2.7 N, and for 90 N desired force, with a force error of about -2.1 N. When the desired force gets closer to the middle range, the results are better.

The experimental setup used for teaching and playback is shown in Figures 8 and 9. Position control is used for the X direction and force control is used for the Z direction. In teaching mode, the load cell information is recorded so that it can be used as the input to the actuator force controller along the Z direction. Simultaneously, the X direction works in friction cancellation mode, so that it can freely slide without adding obvious disturbance to the Z direction load cell from the human operator. The X direction position is also recorded. In playback mode, the X direction position and Z direction force are carried out simultaneously to replay the constrained motion recorded in teaching mode.
A continuous friction model is crucial for the teaching and playback of the X direction position. Since the reference position input in playback mode is generated by the recorded position trajectory in teaching mode, the desired velocity in playback mode can never be ideally zero due to noise and handshaking. If a discontinuous Coulomb friction model is used, the switch of reference velocity around zero (derivative of recorded teaching position) generates noise when the manipulator works close to zero velocity. Furthermore, a continuous friction model can effectively eliminate hysteresis around zero velocity. The friction model is modeled as:

\[
F[v(t)] = F_v v(t) + (F_{c_{\text{pos}}} \text{sat}[\text{sgn}(v(t))] + F_{c_{\text{neg}}} \text{sat}_2[\text{sgn}(v(t))][1 - e^{-(\text{inv}(t)/v_2)}] \text{sgn}(v(t))
\]

where \(v(t)\) is the velocity of the pneumatic actuator, \(F_v\) is the viscous friction force, \(F_{c_{\text{pos}}}\) is the positive direction Coulomb friction, \(F_{c_{\text{neg}}}\) is the negative direction Coulomb friction, \(\text{sat}_1\) and \(\text{sat}_2\) are two saturation functions, \(\text{sgn}\) is the sign function and \(n\) is a constant tunable parameter. The position tracking performance and continuous friction compensation are shown in Figure 10 for square wave input as an illustration of the continuous friction model.

As an illustration, the peg is taught to move in and out of the two holes in a series of random interaction tasks. The operation taught by operator is an arbitrary combination of free space motion and constrained motion with hard surface unpredicted contact. The recorded position trajectory along the X direction and force profile along the Z direction are accurately repeated as shown in Figure 11. The series of constrained interaction tasks are accurately replayed.
amount of energy to its environment. When the control valve for a hydraulic actuator or a pneumatic actuator is off after desired contact force is commanded, both actuators are passive because no internal energy can be produced if the valve is off. So what makes the contact force control so different for a hydraulic system and pneumatic system? The essential difference lies in the media they use. A hydraulic actuator uses incompressible liquid media such as oil, or water combined with soluble oil. A pneumatic actuator uses compressible air. The compressibility of air provides a big advantage when impact occurs. It absorbs the kinetic energy and transforms them into the internal energy of compressed air like the case when a spring hits an unpredicted object. The kinetic energy is transformed into the potential energy of the spring. In the sense of admittance and impedance, a hydraulic piston takes force (pressure difference) as an input and transforms it to a velocity output (volumetric flow rate), therefore hydraulic actuators are admittances. A pneumatic piston takes velocity (mass flow rate) as input and transforms it to a force output (pressure difference), therefore pneumatic actuators are impedances. Admittance-type hydraulic actuators are essentially mismatched for interaction with admittance-type environments. Impedance-type pneumatic actuators are essentially well matched for interaction with admittance-type environment.

One of the most important advantages of contact force control using pneumatic actuator is that when the valve is off, the pneumatic actuator behaves like a nonlinear mass-spring-damper system, which is not only passive but also dissipative. This nonlinear air spring behaves as a vibration absorber simultaneously when the desired contact force is achieved, which makes the contact transition smooth, no oscillation or big overshoot. For hydraulic systems, the contact force has big overshoot and continuous bouncing if the contact is made with a hard surface [12] because kinetic energy of the piston can not be quickly released. And this overshoot and bouncing jeopardizes the stability and robustness of contact force control.

**CONCLUSIONS**

An efficient and intuitive robot programming methodology by directly guiding the robot in the most natural way for force-controlled contact tasks can be achieved by taking advantage of the direct drivability and low natural compliance of a pneumatic manipulator. Pneumatic actuators have the unique feature of natural compliance, which is a great fit to the force control in contact tasks. The robust and stable contact force control of pneumatic actuators has been shown through this study. It can be used in many areas such as assembly and polishing industry.

**REFERENCES**


INSTANTANEOUS OPTIMAL CONTROL OF A PREDATOR-PREY NATURAL RESOURCE WITH A PREY-DEPENDENT FUNCTIONAL RESPONSE

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ABSTRACT

The purpose of this treatment is to develop taxation and management strategies that are broadly applicable to predator-prey natural resource systems. This results in a multi-layered control problem where the control variables are interdependent. One control variable is the rate of taxation. It is assumed that a regulatory agency varies the tax per unit biomass on the harvestable prey to oversee commercial fishing ventures which act to optimize their economic interests. The agency seeks to regulate feasible upper and lower bounds on the protected predator population. A second control variable is taken as a commercial venture’s harvesting effort of the prey. The combined effect of taxation and harvest ensures a cyclic, yet sustainable population of the predator and prey as the taxation rate affects the intensity of harvest. Easy to implement taxation and harvesting effort strategies are derived through the techniques of instantaneous optimal control. Such strategies are particularly useful as they are generated “on-line” as the dynamic system evolves. Furthermore, simulation results are provided to illustrate the economic and ecologic benefits and ramifications of the proposed strategy that result in taxation revenue, revenue for commercial ventures, and sustainable levels of predator and prey biomass.

Keywords: Functional response, instantaneous optimal control, predator-prey natural resource

INTRODUCTION

With expanding human populations placing pressure upon natural resources, it has become imperative to study the sustainability of our natural resources. Perhaps one of the most critical natural resource systems is that of a predator-prey system where the prey is subject to harvest and the predator is protected [1]. In this scenario, the sustainability of predator-prey biomass levels, the interests of commercial ventures which derive their income from the harvest of the prey, and the (likely governmental) regulatory agencies that provide oversight are each significant. Clearly, a balance must be struck between viable population levels for the predator and prey and the interests of commercial ventures with oversight from regulatory agencies. Any regulatory agency must ensure these viable populations, but tax revenue, consistent employment for individuals, and an adequate food supply are realities that must be considered.

The development of harvesting strategies that consider the interests of the populations, the ventures, and the regulatory agency is complex. The literature contains a number of contributions which address this issue. Many authors have addressed challenges surrounding the management of single- and multi-species populations to illuminate reasonable harvesting strategies and the ecologic implications involved. In [2] natural resource management was examined in great depth. The notions of constant yield harvest, fixed effort harvest, and the idea of nonvulnerability of natural resource systems was studied in great detail. Furthermore, local and global stability properties of simple and complex ecosystems were analyzed. Ultimately, optimal control policies that pertain to ecological populations, fisheries, and natural resources were explored.

In [3], the notion of bioeconomic harvesting was examined. This established a relationship between biological and economic equilibrium. Once fiscal and ecological concerns are involved, several mechanisms exist to provide economic benefits to society yet protect nominal levels of predator and prey biomass. To prevent over-fishing of non-protected species restrictions such as catch limits, seasonal restrictions, fees for licensure, property lease fees, and taxation are invoked [1]. It was noted that taxation seems to be an effective mechanism for the regulation of biomass levels. Taxation and maximization of commercial net economic revenue, which ultimately results in the regulation of predator-prey biomass levels will be explored in this treatment.

The nature of the mathematical model which describes predator-prey populations is a non-trivial issue. In this treatment, we wish to focus upon a predator functional response which is prey-dependent, as opposed to ratio-dependent. The predator functional response has received much attention in this regard. The controversy surrounding these two methodologies is addressed in [4]. In this work, we demonstrate that the regulation through taxation is much more “active” in this case of a prey-dependent functional response. While the notion has been proposed that the population dynamics of ratio-dependent models are more interesting [5], for the proposed management strategies, the regulatory agency must act rigorously if the goal is to specify reasonable bounds on predator and ultimately prey biomass levels.
The novelty of this treatment is the manner in which the desired predator and prey biomass levels are realized. The regulatory agency sets the rate of taxation based upon the predator biomass level. The commercial venture(s), quite reasonably, act to maximize their revenue streams. A change in taxation rate actually alters the objective function for the commercial venture. Specifically the net economic revenue for the venture switches with the taxation rate. Therefore, the harvesting effort, from zero to maximum (and anywhere in between) is determined by the venture instantaneously maximizing its net economic revenue at that taxation level. This use of instantaneous optimal control, by construction, yields a harvesting strategy that is independent of many parameters within the predator functional response. What is significant about this is that taxation can regulate prey-dependent systems implicitly based upon a taxation rate chosen as a function of the predator biomass. This allows for tax revenue to be collected by explicitly considering the protected population. Thus, agencies directly preserve protected populations while generating tax revenue. Furthermore, the methodology produces behavior that is typically seen in an optimal control solution and is quite constructive in terms of creation of truly on-line harvesting strategies with an optimization rationale. The numerical simulations will illustrate bang-bang and bang-intermediate (bang-singular) control that is common within optimal control laws. Here, it is felt that replicating optimal behavior through sub-optimal instantaneous optimal control is a contribution worth pursuing. Ongoing research will focus upon issues that would arise during implementation in practice.

With this in mind, we accomplish the following. In the next section, a brief Review of Concepts is presented which contains prior work in the literature related to this treatment. Tax policy regulation of commercial fishing ventures will be one focus [1]. Threshold policy control (harvest) will be a second focus [6]. These areas provide a portion of the theoretical foundation for the current treatment. Furthermore, we relate the proposed algorithm to existing algorithms found in the literature. Following that, the Mathematical Framework is established. This includes the prey-dependent dynamic system model and the optimization framework. Based upon this the harvesting strategy (control law) is derived and we term the algorithm Instantaneous Optimization Net Revenue Harvesting (IONRH). This is a new control algorithm (harvesting strategy) due to its foundations in taxation levels based on predator biomass, coupled with the venture’s move to instantaneously optimize net economic revenue which depends upon taxation rate and prey biomass levels. This results in sustainable, cyclic biomass populations. IONRH is related to additional instantaneous optimal control algorithms; this will be explained in the sub-section IONRH Optimization Framework. In the following section, Algorithmic Performance is studied; examples are presented to illustrate the usefulness of the algorithm. The interplay between taxation levels and control effort will be explored with economic and ecologic impacts discussed. Following this, a Discussion is presented which highlights the novelty of IONRH from both a practical and theoretical point of view. Finally Conclusions are presented.

**REVIEW OF CONCEPTS**

With the stated goal of deriving harvesting effort by considering net economic revenue, we wish to briefly summarize prior results from the literature.

**Bioeconomic Harvesting**

In [1] the authors considered a cost functional that maximized total discounted net economic revenue; these are the “revenue streams” that benefit the individuals (society) represented by the agency. This cost functional was of the form

\[ J = \int_{0}^{\infty} e^{-\delta t} (pq x_1 - C) u dt \]

The state variables are the prey and predator biomass; denoted by \( x_1 \) and \( x_2 \) respectively. The harvesting effort is denoted by \( u \). The parameter \( \delta \) is the instantaneous annual rate of discount, \( p \) is the price per unit biomass of the prey, \( q \) is the catchability coefficient, and \( C \) is the cost of fishing per unit effort.

Considering (1), the harvesting effort specified in [1] was developed in a continuous time form where the time rate of change of harvesting effort was proportional to the commercial venture’s perceived rent. This led to

\[ \dot{u} = \mu [q(p - \tau)x_1 - C]u \]

where \( \tau \) is the tax per unit biomass, \( \mu \) is a parameter that scales the time rate of change of harvesting effort with net economic revenue, and \( (\cdot)' = d(\cdot)/dt \). The expression (2) yielded the harvesting effort, but it was not the control variable. The authors analyzed equilibrium solutions with \( \tau \) the control variable and determined an equilibrium point \((x_1^*, x_2^*, u^*)\) that satisfied optimal control necessary conditions applied to (1). The stability of this equilibrium point was studied. It was determined that the optimal prey and predator biomass were given by \( x_1^* = C/q(p - \tau) \) and \( x_2^* = v x_1^* \). However, the system studied consisted of a ratio-dependent functional response which allowed for equilibrium (constant) prey and predator biomass levels. It is important to note that the term \([u(q(p - \tau)x_1 - C)]\) is the commercial venture’s net economic revenue. Note that in (2), harvesting effort did not cease when net economic revenue was zero; the harvesting effort remained constant. This establishes a precedent that harvest is still feasible even if net economic revenue is zero. The accumulated biomass still retains value, and in fact discounted net economic revenue (1) may increase while net economic revenue is zero. In this treatment we will explore taxation as in [1], but to a system which consists of a prey-dependent functional response. The harvesting strategy will be derived by considering the venture’s net economic revenue, but the discounted net economic revenue will also be monitored.

**Threshold Policy Harvest**

An additional harvesting methodology is known as threshold policy control [6]. The authors develop a “threshold policy with
hysteresis” including placement of bounds on predator and prey populations within appropriate levels. This is done to ensure the health of the ecosystem; bounding trajectories away from regions that may lead to extinction. The authors accomplished this by defining threshold levels where the control would switch between maximum and minimum levels; noting that this could be interpreted as switching between harvest and no harvest. Informally, this switching action would cease harvest if population levels approach a specified lower limit and allow for harvest if populations approach some specified upper limit. The authors accomplished this by defining “virtual equilibrium points”; in effect these equilibrium points attracted trajectories to and from the lower and upper population bounds in a cyclic manner.

This work of [6] is quite relevant to the prey-dependent system in this treatment. In the ratio dependent model, such as that considered in [1], due to its special structure, equilibrium populations may be obtained asymptotically. However, in the prey-dependent case, the predator population will increase when the prey is above its equilibrium value and decrease when the prey population is below this level [7]. This difference in behavior of ratio- and prey-dependent functional responses is due to the nature of their isoclines [7]. Therefore it is unreasonable to attempt to dictate constant equilibria for predator and prey in this case [6]. As will be demonstrated in the following sections, developed algorithms will be used for the prey-dependent systems. Incorporating ideas of threshold policy control [6], a harvesting strategy is developed through which the regulatory agency may adjust taxation rates to produce cyclic biomass behavior, between dictated bounds.

The developed algorithms are of course similar to those of [6] in that they both exhibit switching type behavior on the control to produce reasonable bounds on biomass. However, the proposed methodology is novel in that a change in taxation rate actually alters the objective function (net economic revenue) for the commercial venture. Since the commercial venture harvests based upon instantaneous optimization, this results in a new switching surface that defines whether or not to harvest. This in turn dictates that any harvest will preserve reasonable upper and lower biomass thresholds. Therefore, a two-tiered control problem exists; the agency controls taxation, the venture harvest to maximize economic interests. This methodology provides for consideration of function optimization, taxation, and reasonable biomass levels in a constructive manner.

The proposed IONRH algorithm is of course related to classical control structures in the literature. One such methodology, sliding mode control (SMC), is well known as creating variable structure control laws [8, pg. 291]. In SMC, the control law (or a portion thereof) generally switches (often discontinuously) when the state crosses a surface in the state space (often a line when the dynamic system is defined by two state variables) [9]. Once trajectories reach this reduced order surface, which is typically defined by a linear combination of the state variables set equal to zero, trajectories “slide” or “chatter” towards the origin. It has been demonstrated that SMC is quite robust to disturbance (unknowns) [9]. Additionally, the literature shows examples of sliding surfaces constructed which consider optimization of some performance measure, as in [10]. What is unique about IONRH is that for the two dimensional state equations, the switching surface $\sigma$ is a function of only one state variable, the prey biomass $x_2$ along with biological parameters and the taxation rate. Predator and prey trajectories do not “slide” along the switching surface, rather biomass levels fluctuate between predetermined levels. Then, utilizing threshold policy based on the predator biomass $x_2$ two switching surfaces are created as dictated by the change in taxation rate, which would come as a mandate from the regulatory agency. These characteristics of IONRH will be illustrated in what follows.

**MATHEMATICAL FRAMEWORK**

The mathematical framework considered consists of both differential equations that describe the predator-prey interactions and the relevant function (net economic revenue) which is to be maximized. The harvesting effort which is derived by considering instantaneous optimization is a true compromise. In what follows it is shown that the commercial venture’s interests are considered (through the maximization) yet the regulatory agency ultimately determines appropriate biomass levels where harvest is appropriate. Again, this is in a sense a two tiered control problem.

*Dynamic System Model*

The prey-dependent system considered is given by [7]

$$\dot{x}_1 = r x_1 \left(1 - \frac{x_1}{K}\right) - \frac{\beta x_1 x_2}{\alpha + x_1} - qu x_1$$

(3)

$$\dot{x}_2 = \frac{\beta x_1 x_2}{\alpha + x_1} - \rho x_2$$

where for the prey, $r$ is the intrinsic growth rate, $K$ is the carrying capacity, $\beta$ is the maximum harvesting rate, and $\alpha$ is the half saturation level. For the predator, $s$ is the intrinsic growth rate and $\rho$ is the mortality rate. We assume the harvesting effort is bounded by $0 \leq u \leq u_{max}$.

*IONRH Optimization Framework*

Control laws developed focus upon the cost functional to be instantaneously minimized. This is somewhat related to the techniques of Lyapunov optimizing control which was first developed in [11] with further information and development in [12] and [8, pp. 273-294]. Note that switching surfaces developed by the authors were often based on instantaneous optimization of an objective function. However, as mentioned above, incorporation of multiple switching surfaces (involving prey biomass), determined by taxation rates (determined by predator biomass) makes IONRH unique.

Consider a dynamic system that is a function of the state, the control, and possibly additional parameters or uncertainties. Given the state equations

$$\dot{x} = f(x, u)$$

(4)
and a cost functional to be optimized

\[ J = \int_0^1 f_0(x, u) \, dt \quad (5) \]

the application of optimal control necessary conditions provides the control \( u \) affecting (4) which minimizes (5). However, due to disturbances, the desire to generate a feedback control scheme directly, or other issues, this may not always be practical. Instead, by instantaneous optimal control [8, pp. 273-294] we mean to maximize

\[ J^* = f_0(x, u) \quad (6) \]

at each point along the state trajectory. Of the many choices for the control at each point, the control that instantaneously maximizes the cost \( f_0 \) will be selected. Due to this, \( J \) is often referred to as the accumulated cost and \( J^* \) is referred to as the instantaneous cost. In this treatment, maximization of (1) is of interest for the agency in terms of tax revenue, but the harvesting strategy will be derived from the commercial venture’s net economic revenue. With this in mind, the use of (6) yields

\[ \max_u J^* = \max_u f_0(x, u) \quad (7) \]

The result is that for the dynamic system and cost function of interest (7) the control will be selected via

\[ \max_u [u(q(p - r)x_1 - C)] \quad (8) \]

subject to the state equations (3). Notice that the argument of (8) is the commercial ventures net economic revenue. We prefer to formulate the problem in the spirit of minimization; therefore, the net economic revenue may be maximized by

\[ \min_u [-J^*] = \min_u [-u(q(p - r)x_1 - C)] \quad (9) \]

and so (9) is the working function to be optimized. The integrand of (1) may be interpreted as instantaneous discounted net revenue. Notice that the venture’s objective function (9) is a function of taxation. This is the explicit means by which the taxation and harvesting control variables are linked. Also, the venture’s objective function switches with a change in taxation rate.

A key advantage provided by the instantaneous optimal control methodology is the ease with which feedback control strategies may be developed. It is not necessary to synthesize feedback strategies after application of Pontryagin’s minimum (maximum) principle; feedback harvesting strategies are provided immediately. This is exactly the on-line capability that provides for great ease of implementation. This is a fundamental point; application of Pontryagin’s minimum principle involves adjoint equations [8, pp. 368-372] that are functions of the predator dynamics and therefore the predator functional response. Often, numerical means are required to solve two point boundary problems to satisfy the necessary conditions. Of further note is that the harvesting effort will result in bang-bang and bang-singular control regimes. The difficulty in realizing this behavior was noted in [1]. In this treatment, through instantaneous optimal control, we have produced such behavior and feel that this is significant in view of the difficulties which prevent such behavior from being produced through application of optimal control necessary conditions in many cases.

Minimization of (9) proceeds by forming the Lagrangian and applying first-order necessary conditions [8, pp. 127-128]. Application of the Lagrangian yields

\[ L = [-u(q(p - r)x_1 - C)] - \gamma_1 (u_{max} - u) - \gamma_2 u \quad (10) \]

with \( \gamma_2 \) and \( \gamma_3 \) Lagrange multipliers, where the necessary conditions applied to (10) require

\[ 0 = \frac{\partial L}{\partial u} = -q(p - r)x_1 + \gamma_1 - \gamma_2 \quad (11) \]

\[ 0 \leq (u_{max} - u) \quad (12) \]

\[ 0 \leq u \quad (13) \]

\[ 0 \leq \gamma_1 \quad (14) \]

\[ 0 \leq \gamma_2 \quad (15) \]

\[ 0 = \gamma_1 (u_{max} - u) + \gamma_2 u \quad (16) \]

For simplicity let

\[ \sigma = -(q(p - r)x_1 - C) \quad (17) \]

where from (11)-(16) it is easy to verify that

\[ u = u_{min}, \quad \gamma_1 = 0, \quad \gamma_2 > 0 \quad \text{if } \sigma > 0 \]

\[ u = u_{max}, \quad \gamma_1 > 0, \quad \gamma_2 = 0 \quad \text{if } \sigma < 0 \]

\[ u = u_{opt}, \quad \gamma_1 = \gamma_2 = 0 \quad \text{if } \sigma = 0 \quad (18) \]

Equations (18), along with the agency varying the rate of taxation represents the control strategy IONRH where the commercial venture seeks to maximize its instantaneous net economic revenue. We will illustrate performance as the tax rate is varied. A primary outcome of this section is that given a mandated tax rate per prey biomass, application of (18) produces predator and prey trajectories which are instantaneously optimal in terms of net economic revenue via (10-18). For \( \sigma = 0 \) to be maintained over a finite time interval it is required that \( \sigma = \sigma = 0 \), which implies that

\[ x_1 = 0 \quad (19) \]

resulting in the singular control

\[ u_s = \frac{r(1 - \frac{\sigma}{\beta}) - \frac{\beta x_1}{q(x_1 + x_2)}}{q(x_1 + x_2)} \quad (20) \]

**ALGORITHMIC PERFORMANCE**

The purpose of this section is to demonstrate the utility of the IONRH strategy applied to systems with a prey-dependent functional response as in (3). We choose to demonstrate controlled system behavior through simulation (example) rather than mathematical proofs as it clearly highlights system behavior. This is done to focus on the qualitative behavior of the populations. This analysis requires a more active taxation policy set forth by the regulatory agency which ultimately affects harvest (which is quite active in its own right). This is illustrated through the use of threshold policy harvesting techniques [6], where in the present treatment the taxation rate...
in effect “sets” threshold levels near critical population levels. Here, harvesting effort itself is not regulated; rather the taxation rate placed upon the prey biomass is a function of acceptable upper and lower levels of the predator biomass. To illustrate the capabilities of the harvesting strategy IONRH (18), consider the following example. Note that the simulation results were generated by integrating the relevant state equations with a fourth-order, Runge-Kutta algorithm [8, pp. 21-22] utilizing MATLAB [13].

**Example One (Performance and Revenue Comparison):**

This simulation example uses similar parameters as those in [7]. Net economic revenue will be given by two expressions depending upon the tax rate. One is

\[ J^* = \left[ u(q(p - \tau_h)x_1 - C) \right] = \left[ u(0.1(15 - 10)x_1 - .5) \right] \]

and

\[ J^* = \left[ u(q(p - \tau_l)x_1 - C) \right] = \left[ u(0.1(15 - 2.5)x_1 - .5) \right] \]

where

\[ \dot{x}_1 = .065x_1 \left( 1 - \frac{x_1}{160} \right) - \frac{0.12x_1x_2}{1.2 + x_1} - 0.1ux_1 \]

\[ \dot{x}_2 = \frac{0.8x_1x_2}{1.3 + x_1} - .024x_2 \]

(23)

Values for individual parameters may be easily seen in comparison to (3). Clearly we see that (21) and (22) have differing rates of taxation. To maximize (21) and (22) we execute (18), which results in

\[ \min_u (-J^*) = \min_u [-u(0.1(15 - 10)x_1 - .5)] \]

\[ \min_u (-J^*) = \min_u [-u(0.1(15 - 2.5)x_1 - .5)] \]

and apply (18) with \( u_{\max} = 10 \); trajectories originate from \( x_1(0) = 3 \) and \( x_2(0) = .2 \). Figure 1 contains the predator-prey trajectories as well as the control effort. From this numerical simulation, (18) produced the cyclic trajectories are evident in Fig. 1. Note the multiple regions of singular control \( (0 < u < 10) \) for both the higher and lower threshold levels. These regions provide easy to implement harvesting strategies due to their continuous nature. It is important to note that the threshold levels were set on the predator (protected) population; we will denote a threshold level on the predator by \( \bar{x}_2 \). The threshold levels were realized given \( C = .5, q = .1, p = 15 \), where the agency dictated \( \tau_h = 10 \) and \( \tau_l = 2.5 \) where \( \tau_h \) and \( \tau_l \) are the higher and lower taxation rates. The threshold levels, in a manner similar to [6], were set at \( \bar{x}_2 = .9x_1^* = .9[C/q(p - \tau_h)] = 0.9 \) for the higher taxation rate and \( \bar{x}_2 = 1.1x_1^* = 1.1[C/q(p - \tau_l)] = .44 \) for the lower taxation rate. Time histories of the total discounted net economic revenue (1), with \( \delta = 0.05 \), and the total net economic revenue (24, 25)

\[
\int_0^T J^* = \int_0^T (q(p - \tau)x_1 - C)udt
\]

(26)

and total harvest are presented in Fig. 2. First, due to the exponential in (1) the discounted net economic revenue ultimately approaches an upper bound. Now observe the “step” up in net economic revenue in response to the changing taxation rate. This revenue is gained through the change in taxation which allows for harvest while the predator population is above the lower threshold. Of course, this is followed by a singular control regime; this results in the integrand of (26) equaling zero. While this means that net economic revenue is zero, the venture is still harvesting. Although this is the case, it does not imply that continued harvesting is useless. In fact as mentioned prior it is consistent with alternative harvesting strategies in the literature. The strategy of [1] dictates that the time rate of change of the harvesting effort is zero when net economic revenue is zero (but the harvesting effort itself is not zero). In fact, total discounted net economic revenue may continue to increase. Clearly, society benefits as the discounted net economic revenue increases and the increase in yield provides employment for individuals employed by a commercial venture.

Now suppose the identical simulation is executed with one change; let the instantaneous annual rate of discount \( \delta = .005 \). What has changed is that the time scales between the state equations (23) which represent the evolution of the predator-prey species are allowed to differ significantly from the time scales that describe the instantaneous annual rate of discount. Figure 3 displays time histories of the total harvest, total net economic revenue, and the total discounted net economic revenue. This clearly illustrates the utility of the IONRH strategy; both sustainability of the natural resource and economic concerns for the venture and society may coexist. The total discounted net economic revenue increases despite the fact that net economic revenue is zero during the singular control regimes.

**Example Two (Revenue and Harvesting Effort):**

Net economic revenue will be given by the same two expressions of Example One (21, 22). All parameters remain the same as the first portion of Example One, apart from the maximum harvesting effort. The purpose of this example is to illustrate how harvesting intensity, which may vary due to a single venture increasing/decreasing operations or increased/reduced competing affects revenue streams under IONRH. Therefore we will apply (18) to the state equations (23) while we still maximize (21) and (22) through (24) and (25). Suppose that the maximum harvesting effort is \( u_{\max} = 1.5 \); trajectories originate from \( x_1(0) = 3 \) and \( x_2(0) = .2 \). Figure 4 contains the predator-prey trajectories as well as the control effort. Figure 5 displays time histories of the total harvest, total net economic revenue, and the total discounted net economic revenue. This clearly illustrates how harvesting intensity, which may vary due to a regulatory agency may anticipate biomass levels from known ventures operating and the accumulated catch. Furthermore, ventures immediately gain insight into the financial benefits/repercussions of less/more competition or scaling up or down operations.
DISCUSSION

The analytical developments and numerical simulations provide a basis for a discussion of the utility of the derived taxation and harvesting strategies. Taxation levels were used as a means to influence prey biomass levels. However, this was done implicitly by determining a rate of taxation as a function of predator biomass levels. In Figures 1 and 4, it is clear that such a methodology is successful at maintaining biomass levels between reasonable bounds. Figures 2, 3, and 5 demonstrate the returns that can be expected in terms of net economic revenue and total discounted net economic revenue. Notice that total discounted net economic revenue may increase even when net economic revenue does not. Furthermore, notice in (1) that the total discounted net economic revenue contains a negative exponential term (with $\delta$ positive). Due to this term, the total discounted net economic revenue will approach some finite bound since the integrand of (1) asymptotically approaches zero as $t \rightarrow \infty$. However, changes in interest rates and new harvesting seasons would in effect “re-start” the collection of revenue. Figures 4 and 5 illustrate that under IONRH, both agencies and ventures may anticipate ecologic and economic ramifications of changing conditions. Figures 2 and 5 illustrate that maximum harvesting effort does not provide the greatest returns in terms of discounted net economic revenue. This also holds true for net economic revenue even though the total yield was nearly identical. For the agency, this is vital information; a greater harvesting effort does not drastically increase catch; for a sustainable predator population, this is critical. These results suggest that a regulatory agency may increase tax revenue by restricting harvesting effort. This is an important result if sustainability of the predator is of ultimate concern. For ventures, this may allow a single venture to re-assign its own priorities, not needing to harvest at maximum intensity. However, if additional ventures enter the market, this suggests that competition will negatively impact profit, which is intuitive.

The numerical accuracy and validity of the simulation results from Examples One and Two must be judged to see if the behavior in Figures 1-5 agrees with what is expected analytically. Expected behavior from the state equations (3) and the IONRH control strategy (18), considering total discounted net revenue and discounted net revenue must match the simulation results. Perhaps the easiest manner to see this is by considering Fig. 1. At approximately 130 time units, notice that the predator level reaches the upper threshold $x_2 \approx 9x_2^*$. This caused $a < 0$ resulting in maximum control effort (note that Fig. 1 is “cut” at magnitude 3 so that cyclic behavior may be easily seen) as dictated by (18). The prey biomass is quickly affected, and the control effort drops and assumes singular control until approximately 275 time units. Note that for figures 1-5, 250,000 time steps were distributed over the 750 time units. To further judge the numerical accuracy of the algorithm, consider Fig. 6. This figure was created by using the same parameters as in Example One. The difference is that the time interval was truncated to 400 time units and 150,000 time steps. Again, the magnitude scale was decreased to focus upon predator and prey trajectories. If the algorithm is accurate, singular control should correspond with instances of $x_1 = \text{constant}$ and the harvesting effort should switch in coordination with predator levels of $\bar{x}_2 = 0.9$ and $\bar{x}_2 = 0.44$. This behavior is clearly illustrated in Fig. 6. A final consideration is the sensitivity of the algorithm. Numerically, it the conditions $\bar{x}_2 = 0.9$ and $\bar{x}_2 = 0.44$ must be detected accurately if the mandated bounds on predator population are to be realized. Integration step sizes that are too large would result in less accurate results “overshooting” the conditions proportional to the step size. Step sizes that are unreasonably small, would result in much greater cost in terms of computation. However, in terms of future work, it is felt that the numerical accuracy/sensitivity will be much less of an issue than challenges faced implementing the algorithm in practice. It is this next step that we will focus upon in upcoming research.

CONCLUSIONS

In this treatment, the IONRH control algorithm was proposed to provide for reasonable bounds on predator and prey biomass for systems with a prey-dependent functional response. The derived strategies utilize instantaneous optimal control techniques for several critical reasons. Taxation rates were based strictly upon predator biomass levels. Harvesting strategies, in reaction to the taxation rate were based upon economic factors and prey biomass levels. Built upon the ideas presented in [6], but incorporating instantaneous optimization and taxation rates, we were able to show numerically that predator-prey levels may be maintained within upper and lower bounds, even with uncertainty. Policy makers may actively dictate these upper and lower bounds through taxation. Commercial ventures have a clear idea as to potential revenue streams and decisions to scale up or down operations. Economic ramifications for an increase or decrease in competition between ventures may also be estimated. The common thread throughout is the optimization based rationale for all decisions.

REFERENCES


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Figure 1. Biomass and Harvesting Trajectories for Example One

Figure 2. Economic Returns and Total Yield for Example One

Figure 3. Economic Returns and Total Yield for Example One
Figure 4. Biomass and Harvesting Trajectories for Example Two

Figure 5. Economic Returns and Total Yield for Example Two

Figure 6. Biomass and Harvesting Trajectories for Example One
**INVESTIGATION INTO THE SURVIVABILITY OF AN IRRADIATION BASIN LINER SUBJECTED TO IMPACT LOADING RESULTING FROM THE HYPOTHETICAL 30 FOOT DROP OF A SHIPPING CASK**

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**ABSTRACT**

Of interest was the survivability of a 7 foot diameter by 14 foot deep, water-filled, stainless steel-lined irradiation basin given a hypothetical drop impact scenario. A 5,443 kg (12,000 lb) type B radioactive shipping cask was to be lowered by crane into the basin, through a 30 foot high roof opening. The primary question was whether, in the event of an accidental drop, the basin walls and floor would rupture upon cask impact thereby allowing radiation-contaminated water to leak into the surrounding soil.

Using the explicit finite element (FE) solver, MSC/MD Nastran, numerical simulations were performed to: 1) validate the full scale FE technique employed by comparing the basin floor deformation of reduced-scale FE models to physical models, 2) estimate the velocity of the cask through water, using coupled Eulerian and Lagrangian FE mesh formations to address the fluid-structure interaction (FSI), and 3) determine whether basin liner rupture would result from impact loading for the four cask drop orientations considered. The investigative results failed to eliminate the possibility of basin liner rupture given a specific cask drop orientation. Ultimately, the cask was not lowered into the basin.

**Keywords:** Shipping Cask, FSI, Simulation, Fluid Impact, Finite Element Analysis

**INTRODUCTION**

An investigation was conducted into whether a 12,000 pound cask would rupture the stainless steel liner of a water-filled basin used to store radioactive material if the cask was accidentally dropped into the basin. This investigation formed part of larger risk assessment conducted by a customer attempting to determine the feasibility of lowering the cask into the basin, loading it with the radioactive materials, and then lifting the cask out of the basin. Rupture of the liner at any time during this operation would result in contaminated water leaking into the surrounding soil. Thus, an estimate of worst-case damage was required.

A cross-sectional representation of the basin, whose installation dates from the 1950's, is shown in Figure 1. While a number of details regarding the basin's construction were not known, it was known that the innermost wall and floor (referred to as the liner) were constructed from 1/2 inch thick 304 stainless steel (304SS). The vertical wall of the liner had a jacket of 1/4 inch thick 304SS spaced 1 inch outward. Outside this jacket was a layer of sand and corrugated steel and finally the surrounding soil. The floor of the liner was welded to the vertical wall of the liner and was slightly domed in shape. Underneath the floor was approximately 30 inches of concrete as well as at least two concrete anchors of undetermined design. The investigation at issue considered only the liner and its survivability. Any structural contributions from the jacket, sand, corrugated steel, anchors and soil were not considered.

![Figure 1. Cross-Section of Storage Basin](image-url)
A cross-sectioned CAD model of the type B shipping cask under consideration is shown in Figure 2. Used to transport radioactive material, it was cylindrical in shape, with an outside diameter and height of 30 inches and 40 inches, respectively. It consisted of an outer and inner shell fabricated from stainless steel with the void between the two shells cast with high purity lead to provide radiation shielding for the cask contents. A lid could be inserted into a stepped opening in the top and bolted in place. The approximate weight of this cask including payload was 12,000 pounds. Integral cask impact limiters were not considered in this investigation.

**METHODOLOGY**

The assumption was made that the worst-case damage to the basin liner would occur from the cask striking the domed basin floor as opposed to striking the top edge of the inside liner or glancing off the wall while sinking. The validity of this assumption was confirmed by a series of numerical simulations not discussed in this paper.

Due to the presence of multiple nonlinearities stemming from fluid structure interaction (FSI), high-speed impact and the potential for large deformations, the numerical calculations were performed using the finite element (FE) technique. The software package used was the FE simulation software MSC/MN Nastran [1, 2] which contained the explicit solver, Solution 700 (a combination of LS-Dyna and MSC.Dytran) – a solver well suited for use on problems involving impact and fluid structure interaction [3].

As validation of the approach taken by numerical simulation, experimentation was performed using a simplified reduced-scale basin liner and cask model. The deformation resulting from this experimental testing was then compared to the results taken from an FE simulation of the same reduced-scale setup. Good agreement between experimental and simulation results would be necessary to establish with a reasonable degree of engineering certainty that numerical simulation could be employed to investigate the impact response of the full-scale basin.

**REDUCED-SCALE MODEL EXPERIMENTATION**

Simplified reduced-scale models of the cask and basin were fabricated. Due to material availability and fabrication limitations, the reduced-scale cask and basin as fabricated had a scale factor of approximately 0.37. The simplified basin model was fabricated from 3/16" thick ASTM A36 steel and consisted of a rolled and welded cylinder and welded in domed floor with uniform curvature appropriately scaled down but similar to that of the full scale basin floor shown in Figure 1. This basin model was anchored to a thick concrete slab and filled completely with water. Only the bottom perimeter of the basin was supported by concrete. The concrete did not fully support the domed floor as was the case with the actual full-scale basin.

A simplified cask model was fabricated consisting of a welded 1/4 inch thick outer ASTM A36 steel shell that was filled with lead. To simplify fabrication, the cask model did not include a lid or inner steel shell. The cask model weighed 384 pounds including a lifting eye attached to the top to facilitate dropping.

To conduct the drop tests a custom release mechanism was designed and fabricated that minimized the rotation and lateral movement of the cask at the moment of release. The mechanism accomplished this by reducing the slight side forces that apparently caused unpredictable results in some early test drops where a commercially available drop hook was used. With the release mechanism attached to a forklift, the cask was dropped from a height of 15 feet above the water surface – a height arbitrarily chosen but estimated to inflict measurable damage on the basin floor based on preliminary testing performed using another basin.

Several finite element (FE) models were created and are shown in Figure 4. The first was an FE model of the reduced-scale experimental setup. To replicate the concrete conditions
of the experimental setup, the concrete slab modeled under the reduced-scale basin model was flat and supported the basin only at the perimeter. The second was a complete full-scale FE model built representing the subject full-scale basin liner. This model (as well as the reduced-scale FE model) had a stationary Eulerian mesh in which the water or air flowed while conserving mass, momentum and energy. The primary use of this second model was to estimate the impact speed with which the cask struck the basin floor for a given orientation. The third FE model was a partial full-scale FE model created for estimating degree of damage to the basin floor due to the impact of the cask for various orientations. To simplify modeling and significantly reduce solver runtime this third FE model did not include water. This was considered a conservative modeling choice as a layer of fluid (in this case, water) in between a projectile (the cask) and a target (the floor) provides a cushion reducing the impact force [4]. Both of the full-scale models had concrete fully supporting the domed basin floor.

As a conservative assumption, the model of the full-scale basin included only the 1/2 inch 304SS and none of the surrounding jackets, corrugated steel, sand or soil. The liner was modeled with CQUAD4 shell elements that were assigned a thickness of 1/2 inch. The mid-plane of elements forming the floor were spaced one-half the shell thickness away from the concrete elements. For both the reduced- and full-scale basins, a linear-plasticity material model (MATD024) with a bilinear stress-strain response was assumed [5, 6]. The dynamic properties assumed are shown in Figure 5. Conservatively, the maximum elemental strain permitted for A36 and 304SS before failure (removal from calculation) occurred was 0.2 and 0.4 respectively [7].

The concrete under the basin floor was modeled using CHEXA8 brick elements. All nodes on the bottom face of the concrete were given rigid constraints. In reality, some of the shock seen by the concrete would be dissipated down into the supporting soil; thus, this constraint was considered a conservative modeling choice. The material model used for the concrete was the LS-DYNA concrete Model 159. Since no documentation of the existing concrete under the basin was available, the default values of 2320 kg/m³ for density, 30 MPa for compressive strength, $f'$ and 19 mm for concrete aggregate size were used [8, 9].

The reduced-scale and full-scale casks were modeled using a single layer of rigid shell elements and assigned thicknesses of 1/4 and 3/8 inch, respectively. The density assigned to the rigid material model (MATD020) was set to a value resulting in a reduced-scale and full-scale cask weight of 384 and 12,000 pounds, respectively. The use of a rigid material model was considered a conservative modeling choice as it resulted in greater impact energy being imparted to the basin liner.

**Figure 4. Finite Element Models: a) Reduced-Scale Model with 11° Tilt b) 37° Tilt Full-Scale Model with Euler Mesh c) Partial Full-Scale Model with 37° Tilt**

As a conservative assumption, the model of the full-scale basin included only the 1/2 inch 304SS and none of the surrounding jackets, corrugated steel, sand or soil. The liner was modeled with CQUAD4 shell elements that were assigned a thickness of 1/2 inch. The mid-plane of elements forming the floor were spaced one-half the shell thickness away from the concrete elements. For both the reduced- and full-scale basins, a linear-plasticity material model (MATD024) with a bilinear stress-strain response was assumed [5, 6]. The dynamic properties assumed are shown in Figure 5. Conservatively, the maximum elemental strain permitted for A36 and 304SS before failure (removal from calculation) occurred was 0.2 and 0.4 respectively [7].

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**Figure 5. Bilinear Stress-Strain Models and Material Properties Used for 304SS & A36 Steel**

The reduced-scale and full-scale casks were modeled using a single layer of rigid shell elements and assigned thicknesses of 1/4 and 3/8 inch, respectively. The density assigned to the rigid material model (MATD020) was set to a value resulting in a reduced-scale and full-scale cask weight of 384 and 12,000 pounds, respectively. The use of a rigid material model was considered a conservative modeling choice as it resulted in greater impact energy being imparted to the basin liner.
To model the water filling basins and the atmospheric air, a fixed Eulerian mesh was created that had as its boundaries the rigid exterior of the moving cask, the inner basin liner, and a "dummy" boundary surrounding the space immediately above the basin shown in Figure 6. At the start of the numerical simulation Euler elements above the water level were initialized with the ideal gas properties of atmospheric air, and the elements within the basin were initialized with the fluid properties of water.

Glued contact was setup between the bottom perimeter nodes of the basin and the top face of the concrete elements. SOL700's default parameters for adaptive contact were used for contact between the concrete and the basin floor and cask.

\[ v_{\text{water}\text{impact}} = \sqrt{2 \cdot g \cdot H} = \sqrt{2 \cdot \left(32.2 \frac{ft}{s^2}\right) \cdot 15 ft} = 31.1 \frac{ft}{s} \]

Figure 7. Reduced-Scale Simulation a) Basin Deformation & Von-Mises Stress Distribution b) Cask Velocity

The resulting basin floor deformation that occurred in the scale model simulation was compared to the experimental results. Figure 8 shows a side view comparison illustrating the good agreement observed in terms of shape and amount of deformation. Following the impact, the peak basin material strain indicated by FE results was 0.07.
b.

Figure 8. Comparison of Simulation & Experimental Scale Model Basin Floor Deformation

STUDY OF FULL SCALE BASIN

As discussed, good agreement was noted between the deformation resulting from numerical simulation and physical experimentation using a reduced-scale model. This agreement provided verification of the numerical simulation software output and reasonably justifies usage of numerical simulation to study the behavior of the full-scale basin.

Because it was unknown what orientation a cask would have just before it strikes the water following an accidental drop, several representative orientations and positions were considered. Figure 9 shows these orientations: a) horizontal, b) vertical, c) off-center vertical and d) 37° oblique. The oblique angle of 37° was used because this places the cask's center of gravity above the corner of impact [11].

Analysis of the full-scale basin was broken into two steps. The first step was to simulate a drop of the cask at each of the cask orientations shown in Figure 9 through the full depth of water and allow it to sink towards the basin floor. Using equation 1, the initial velocity assigned to the cask just prior to water entry following a 30 ft fall above the water was found to be 43.9 ft/s (13.4 m/s). Figure 10 shows a typical water splash as calculated by numerical simulation for the vertical cask orientation. It is noted that the shape of the developed splash and the formation of the air cavity agrees well with prior projectile-fluid entry research and experimentation [12].

Just before the cask struck the floor, the velocity was noted and used as an input for the second simulation. This step used the complete full-scale basin model and included water. The second step was to place the cask approximately 1 inch from the bottom of basin floor and set the cask's initial velocity (e.g. 40.7 ft/s or 12.4 m/s in the case of the oblique orientation) to the velocity just before impact noted in the first step. This step used the partial full-scale basin model that, as mentioned before, did not include water.

Figure 10. Water Splash and Cavity Formation for Vertical Cask Orientation

PREDICTED BASIN RESPONSE BASED ON FINITE ELEMENT APPROACH

For each of the aforementioned cask orientations, numerical simulations of the cask striking the basin floor were run. Figure 11 shows the Von-Mises stress distribution in the basin liner and concrete just prior to cask rebound.
Drops in the horizontal, vertical and off-center vertical orientations resulted in low elemental strain levels, such that failure was not predicted to occur. However, the 37° oblique drop resulted in basin elements near the point of cask corner contact exceeding the allowable strain – these elements were removed from the calculation by the solver. Additionally, Figure 11(d) shows some concrete element failure also occurred under the region of basin liner rupture.

CONCLUSION
Based on the foregoing results, it was concluded that, if the 12,000 pound cask is conservatively treated as a rigid body, basin liner rupture could occur given certain cask orientations – specifically the 37° oblique orientation. Because of this, the final report could not eliminate the possibility of basin rupture even though the hypothetical scenario under which it could occur (the cask being accidentally dropped by a crane) was an unlikely event.

It should be noted, however, that some work was performed which treated the cask as a deformable body, which allowed the cask to absorb energy by plastically deforming upon impact. When this assumption was made, avoidance of basin rupture may have been possible even for the worst-case oblique drop as shown in Figure 12.

Nevertheless, the customer ultimately elected to not lower a cask into the basin and instead removed the water, filled the entire basin (including its radioactive contents) with grout and extracted it by crane.

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REFERENCES


PASSIVE VARIABLE STIFFNESS SUSPENSION SYSTEM

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ABSTRACT
This paper presents the design and analysis of a passive variable stiffness mechanism. The mechanism is based on a recently designed variable stiffness mechanism. It consists of a horizontal control strut and a vertical strut. The main idea is to vary the load transfer ratio by moving the location of the point of attachment of the vertical strut to the car body. This movement is controlled passively using the horizontal strut. The system is analyzed using an $L_2$-gain analysis based on the concept of energy dissipation. The analysis, and subsequent simulation results, show that the variable stiffness suspension achieves better performance than the constant stiffness counterpart. The performance criteria used are; ride comfort, characterized by the car body acceleration, suspension deflection, and road holding, characterized by tire deflection.

NOMENCLATURE
$y_u$ Vertical displacement of the unsprung mass.
$y_s$ Vertical displacement of the sprung mass.
$h_u$ Half distance between points C and D.
$l_s$ Vertical strut length.
$l_{0_s}$ Natural length of vertical strut.
$l_D$ Length of the lower wishbone.
$H$ Height of the control mass from the pivot point of the lower wishbone.
$x$ Distance between points O and A along the lower wishbone.
$k_s, b_s$ Tire spring constant and damping coefficient.
$k_v, b_v$ Vertical Strut stiffness and damping coefficient
$k_u, b_u$ Control (Horizontal) Strut stiffness and damping
$m_s, m_u, m_d$ Sprung, unsprung and control masses.
$I_c$ Moment of inertia of control arm.

$\lambda_{\text{min}} \{A\}$ The minimum eigenvalue of the matrix $A$
$\lambda_{\text{max}} \{A\}$ The maximum eigenvalue of the matrix $A$
$A_{i,j,k,l}$ The sub-matrix of matrix $A$ formed by rows $i$ to $j$ and columns $k$ to $l$
$A_{i,j}$ The sub-matrix of matrix $A$ formed by rows $i$ to $j$ and all columns
$\text{tr}\{A\}$ The trace of the matrix $A$
$\text{det}\{A\}$ The determinant of the matrix $A$
$L_s(q_1, q_2)$ The set of points that lie on the line segment joining the vectors $q_1$ and $q_2$
$I$ Identity matrix
$e_{i,n}$ The $i$th column of the identity matrix of dimension $n$
$\|v\|$ The Euclidean norm of the vector $v$
$\mathbb{R}$ The set of real numbers
$\text{Re}\{\alpha\}$ The real part of the complex number $\alpha$

1 INTRODUCTION
Improvements over passive suspension designs is an active area of research [2, 27, 6, 19, 21, 20, 11, 13, 12, 8, 23]. Past approaches utilize one of three techniques [4], adaptive[9], semi-active[7, 6] or fully active suspension[9, 26]. An adaptive suspension utilizes a passive spring and an adjustable damper with slow response to improve the control of ride comfort and road holding. A semi-active suspension is similar, except that the adjustable damper has a faster response and the damping force is controlled in real-time. A fully active suspension utilizes a passive spring and an adjustable damper with slow response to improve the control of ride comfort and road holding. A semi-active suspension is similar, except that the adjustable damper has a faster response and the damping force is controlled in real-time. A fully active suspension replaces the damper with a hydraulic actuator, or other types of actuators such as electromagnetic actuators, which can achieve optimum vehicle control, but at the cost of design complexity, expense, etc. The fully active suspension is also not fail-safe in the sense that performance degradation results whenever the control fails, which may be due to either mechanical, electrical, or software failures. Recently, research in semi-active
suspensions has continued to advance with respect to capabilities, narrowing the gap between semi-active and fully active suspension systems. Today, semi-active suspensions (e.g. using Magneto-Rheological (MR)[4], Electro-Rheological (ER)[18] etc) are widely used in the automobile industry due to their small weight and volume, as well as low energy consumption compared to purely active suspension systems.

However, most semi-active systems are designed to only vary the damping coefficient of the shock absorber while keeping the stiffness constant. Meanwhile, in suspension optimization, both the damping coefficient and the spring rate of the suspension elements are usually used as optimization arguments. Therefore, a semi-active suspension system that varies both the stiffness and damping of the suspension element could provide more flexibility in balancing competing design objectives.

This paper presents the design and analysis of a passive variable stiffness suspension system. The design is based on existing stiffness variation concepts. Knaap et al.[22, 25, 8] designed a variable geometry actuator for vehicle suspension called the Delft active suspension (DAS). Although, the intention of the design was not to vary the stiffness of the suspension system, the design used a variable geometry concept to vary the suspension force by effectively changing the stiffness of the suspension system. The basic idea behind the DAS concept is based on a wishbone which can be rotated over an angle and is connected to a pretensioned spring at a variable location. The spring pretension generates an effective actuator force, which can be manipulated by changing the position. This was achieved using an electric motor. Recently, Anubi et al.[3] came up with a design of a variable stiffness mechanism. The mechanism, which is a simple arrangement of two springs, a lever arm, and a pivot bar, has an effective stiffness that is a rational function of the horizontal position of the pivot. The effective stiffness is varied by changing the position of the pivot while keeping the point of application of the external force constant. The variation of concept used in this paper uses the “reciprocal actuation” concept in [3] to effectively transfer energy between the vertical traditional strut and a horizontal oscillating control mass, thereby improving the energy dissipation of the overall suspension system.

The rest of this paper is organized as follows. In Section 2, the variable stiffness concept is described, and the variable stiffness suspension system introduced. A detailed analysis of the system is presented in Section 3. Time domain and frequency domain simulation results are presented in Section 4. The conclusion follows in Section 5.

2 SYSTEM DESCRIPTION

This section gives a detailed description of the variable stiffness concept, overall system, and dynamic modeling of the system.

2.1 Variable Stiffness Concept

The variable stiffness mechanism concept is shown in Figure 1. The lever arm OA, of length \( L \), is pinned at a fixed point O and free to rotate about O. The spring AB is pinned to the lever arm at A and is free to rotate about A. The other end B of the spring is free to translate horizontally as shown by the double headed arrow. Without loss of generality, the external force \( F \) is assumed to act vertically downwards at point A. \( d \) is the horizontal distance of B from O. The idea is to vary the overall stiffness of the system by letting \( d \) vary passively under the influence of the horizontal strut. Let \( k \) and \( l_0 \) be the spring constant and the free length of the spring AB respectively, and \( \Delta \) the vertical displacement of the point A. The overall free length \( \Delta_0 \) of the mechanism is defined as the value of \( \Delta \) when no external force is acting on the mechanism.

2.2 Mechanism Description

The suspension system considered is shown in Figure 2, and the schematic diagram is shown in Figure 3.
Figure 3: Quarter Car Model

The model is composed of a quarter car body, wheel assembly, two spring-damper systems, road disturbance and a lower wishbone. The points OAB is the same as shown in the variable stiffness mechanism of Figure 1. The horizontal control force \( u \) controls the position \( d \) of the control mass \( m_d \) which, in turn, controls the overall stiffness of the mechanism. The tire is modeled as linear spring of spring constant \( k_r \).

The assumptions adopted in Figure 3 are summarized as follows:

1. The lateral displacement of the sprung mass is neglected, i.e only the vertical displacement \( y_s \) is considered.
2. The wheel camber angle is zero at the equilibrium position and its variation is negligible throughout the system trajectory.
3. The springs and tire deflections are in the linear regions of their operating ranges.

2.3 Equations of Motion

Let

\[ q = \begin{bmatrix} y_s \\ \theta \\ d \end{bmatrix}, \quad (1) \]

be defined as the generalized coordinates. The equations of motion, derived using Lagrange’s method, are then given by

\[ M(\theta)\ddot{q} + C(\theta, \dot{\theta}) + B(\theta)\dot{q} - K(q) + G(\theta) = e_3 u + W_d(\theta)d, \quad (2) \]

where

\[ M(\theta) = \begin{bmatrix} m_s + m_u + m_d & m_l D \cos \theta & 0 \\ m_l D \cos \theta & I_c + m_l l_D^2 \cos^2 \theta & 0 \\ 0 & 0 & m_d \end{bmatrix}, \]

\[ C(\theta, \dot{\theta}) = -m_l l_D \dot{\theta}^2 \sin \dot{\theta} \omega(\theta), \]

\[ w(\theta) = \begin{bmatrix} 1 \\ l_D \cos \theta \\ 0 \end{bmatrix} \]

\[ B(\theta) = \begin{bmatrix} b_t & b_l D \cos \theta & 0 \\ b_l D \cos \theta & b_l D^2 \cos^2 \theta + b_s g \theta & b_s g_d \theta \\ 0 & b_s g_d & b_s g_d \end{bmatrix}, \]

\[ g_d(d, \theta) = \frac{(d - x \cos \theta)^2}{H^2 + d^2 + x^2 - 2xd \cos \theta - 2Hx \sin \theta}, \]

\[ g_d(d, \theta) = \frac{2x(d - x \cos \theta)(d \sin \theta - H \cos \theta)}{H^2 + d^2 + x^2 - 2xd \cos \theta - 2Hx \sin \theta}, \]

\[ g_d(d, \theta) = \frac{x^2(d \sin \theta - H \cos \theta)^2}{H^2 + d^2 + x^2 - 2xd \cos \theta - 2Hx \sin \theta}, \]
\[
K(q) = \begin{bmatrix}
    k_1 (\rho_1 - 1)(y_s + l_D \sin \theta) \\
    k_1 (\rho_1 - 1)y_D \cos \theta (y_s + l_D \sin \theta) \\
    k_s (\rho_s - 1)(d - x \cos \theta)
\end{bmatrix}
+ \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\]

\[
G(\theta) = \begin{bmatrix}
    m_s + m_u + m_d \\
    m_u l_D \cos \theta \\
    0
\end{bmatrix}
g,
\]

\[
W_d(\theta) = \begin{bmatrix}
    k_1 (\rho_1 - 1) & b_i \\
    k_1 l_D (\rho_1 - 1) \cos \theta & b_i l_D \cos \theta \\
    0 & 0
\end{bmatrix},
\]

\[
d_c = \begin{bmatrix}
    r \\
    \dot{r}
\end{bmatrix}.
\]

\(r(t)\) is the road signal, and \(\rho_s\) and \(\rho_i\) characterize the compression of the vertical strut and tire springs respectively. They are defined as the ratio of the free length and instantaneous length of the corresponding spring.

### 2.3.1 Properties

The following properties of the dynamics given in (2) are exploited in subsequent analyses:

1. The inertia matrix \(M(\theta)\) is symmetric, positive definite. Also, since each element of \(M(\theta)\) can be bounded below and above by positive constants, it follows that the eigenvalues, hence the singular values of \(M(\theta)\) can also be bounded by constants. Thus, there exists \(m_1, m_2 \in \mathbb{R}^+\) such that

\[
\begin{align*}
    m_1 \|x\|^2 &\leq x^T M(\theta) x \leq m_2 \|x\|^2, \quad \text{and} \\
    \frac{1}{m_2} \|x\|^2 &\leq x^T M^{-1}(\theta) x \leq \frac{1}{m_1} \|x\|^2, \quad \forall x \in \mathbb{R}^2
\end{align*}
\]

2. \(C(\theta, \dot{\theta})\) can be upper bounded as follows

\[
\|C(\theta, \dot{\theta})\| \leq c_1 \dot{\theta}^2, \quad c_1 \in \mathbb{R}^+.
\]

Also, there exist a matrix \(V_m(\theta, \dot{\theta})\) such that

\[
C(\theta, \dot{\theta}) = V_m(\theta, \dot{\theta}) q
\]

\[
x^T \left( \frac{1}{2} M(\theta) - V_m(\theta, \dot{\theta}) \right) x = 0, \quad \forall x \in \mathbb{R}^2
\]

The property in (6) is the usual skew symmetric property of the Coriolis/centripetal matrix of Lagrange dynamics [14].

3. The damping matrix \(B(\theta)\) is symmetric and positive semi definite. Also, there exists positive definite matrices \(B\) and \(\overline{B}\) such that

\[
0 < x^T Bx \leq x^T \overline{B} x \leq x^T \overline{B} x, \quad \forall x \in \mathbb{R}^2.
\]

4. The stiffness vector \(K(q)\) is Lipschitz continuous, i.e there exists a positive constant \(k_2\) such that

\[
\|K(q_1) - K(q_2)\| \leq k_2 \|q_1 - q_2\|.
\]

5. The unique static equilibrium point \(q_0 = \left[ y_{s0}, \theta_0, d_0 \right]^T\) of the undisturbed system is known and is given by

\[
K(q_0) - G(\theta_0) + e_{33} u_0 = 0.
\]

### 2.4 Performance Objective

As usual with suspension systems designs, the performance criterion is expressed in terms of the ride comfort, suspension deflection, and dynamic tire force. The performance vector

\[
z = \begin{bmatrix}
    \omega_1 y_{eba} \\
    \omega_2 y_{sd} \\
    \omega_3 y_{dtf}
\end{bmatrix}
\]

characterizes the ride comfort, suspension deflection, and road holding performances, where \(\omega_1, \omega_2,\) and \(\omega_3\) are the respective performance weights for car body acceleration \(y_{eba}\), suspension deflection \(y_{sd}\), and dynamic tire force \(y_{dtf}\). The unforced disturbance-free version of the dynamics in (2) is given by

\[
M(\theta) \ddot{q} = -C(\theta, \dot{\theta}) - B(\theta) \dot{q} + K(q) - K(q_0)
\]

\[
+ e_{33} u_0 - G(\dot{\theta}) + G(\theta_0) - e_{33} u_0.
\]

\[=-C - B\dot{e} - \dot{Ke},
\]

where...
Using the Cauchy-Schwarz inequality, \( y_{\text{def}}(t) \) can be upper bounded as
\[
y_{\text{def}} \leq \sqrt{1 + k_2^2} \| e \|
\] (21)
Finally, the performance vector in (10) can be upper bounded as
\[
\| e \|^2 \leq \phi_1 (\| e \|^2 + \| e \|^2) + \phi_2 \| e \|^2
\] (22)
where
\[
\phi_1 = \omega_1^2 \left( \frac{c_1 (c_1 + k_2)}{m_1^2} \| e \|^2 + \frac{2 c_1 b_2}{m_1^2} \| e \|^2 + \frac{b_2 (b_2 + k_2)}{m_1^2} \right)
\phi_2 = \omega_2^2 k_2 (k_2 + c_1 + b_2) + \omega_2^2 k_2 + \omega_2^2 (k_2 + 1)
\]

3 SYSTEM ANALYSIS

This section presents the finite-gain stability analysis of the system based on the dynamics in (2), and the performance vector in (10). The disturbance \( d \) is assumed to be unknown a priori but bounded in the sense that \( d \in L_2 \). As a result, robust optimal control is considered in which the gain of the system is optimized under worst excitations [5, 10, 16, 24]. The following definition describes the notion of stability used in the subsequent analyses.

Finite-Gain \( \mathcal{L} \)-Stable [24] Consider the nonlinear system
\[
x = f(x, w)
\]
\[
z = h(x)
\] (23)
where \( x \in \mathbb{R}^n, w \in \mathbb{R}^p, z \in \mathbb{R}^m \) are the state, input, and output vectors respectively. The system in (23), with the mapping \( M_H : L_2^z \to L_2^w \), is said to be finite-gain \( \mathcal{L} \)-stable if there exist real constants \( \gamma, \beta \geq 0 \) such that
\[
\| M_H (w) \| \leq \gamma \| w \| + \beta,
\] (24)
where \( \| . \| \) denotes the \( \mathcal{L} \) norm of a signal, and \( L_2^z \) is the extended \( \mathcal{L} \) space defined as
\[
L_2^z = \{ \chi \mid \chi \in L^z, \forall \tau \in [0, \infty) \}
\] (25)
where \( \chi \) is a truncation of \( \chi \) given as
\[
\chi(t) = \begin{cases} \chi(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}
\] (26)
For the purpose of this paper, the $L_2$-space is considered, hence the finite-gain $L$-stability condition in (24) is rewritten as:

$$\|M_H(w)\|_2 \leq \gamma \|w\|_2 + \beta,$$

(27)

where $\|\|$ denotes the $L_2$ norm of a signal given by

$$\|x\|_2 = \left( \int_0^\infty x^T(t)x(t)dt \right)^{\frac{1}{2}}.$$  

(28)

$\gamma^* = \inf \left\{ \gamma : \|M_H(w)\|_2 \leq \gamma \|w\|_2 + \beta \right\}$ is the gain of the system, and, in the case of linear quadratic problems, is the $H_\infty$ norm of the system. Given an attenuation level $\gamma > 0$, and the corresponding system dynamics, the objective is to show that (27) is satisfied for some $\beta > 0$. This solution is approached from the perspective of dissipative systems [5, 24].

The following definition describes the concept of dissipativity with respect to the system in (23)

**Dissipativity**

The dynamics system (23) is dissipative with respect to a given supply rate $s(w, z) \in \mathbb{R}$, if there exists an energy function $V(x) \geq 0$ such that, for all $x(t_0) = x_0$ and $t_f \geq t_0$,

$$V(x(t_f)) \leq V(x(t_0)) + \int_0^{t_f} s(w, z)dt, \quad \forall w \in L_2.$$  

(29)

If the supply rate is taken as

$$s(w, z) = \gamma^2 \|w\|^2 - \|z\|^2,$$  

(30)

then the dissipation inequality in (29) implies finite-gain $L$-stability [24], and the system is said to be $\gamma$-dissipative. The dissipativity inequality is then written as

$$\dot{V} \leq \gamma^2 \|w\|^2 - \|z\|^2.$$  

(31)

In the subsequent development, two stiffness variation cases are considered; Constant Case and Passive Case. In each case, it is assumed that the closed loop system

$$M_1(\theta)q_1 + C_1(\theta, \dot{\theta}) + B_1(\theta)q_1 - K_1(q_1) + G_1(\theta) = w,$$  

(32)

where

$$M_1 = M_{1,2,1,2}, \quad C_1 = C_{1,2},$$

$$K_1 = K_{1,2}, \quad B_1 = B_{1,2,1,2},$$

and $w = W_{\theta_1}d_\theta$, $W_{\theta_1} = W_{d,1,2,1,2}$

is $\gamma$-dissipative with respect to the supply rate

$$s(w, z) = \gamma^2 \|w\|^2 - \|z\|^2.$$  

(33)

### 3.1 Case 1: Constant Stiffness

For this case, the control mass is locked at a given position $d$. As a result, the overall stiffness is constant for the entire trajectory of the system. For this case, the dynamics in (2) reduces to (32), in which the corresponding dynamics of the control mass has been eliminated. Let

$$e_1 = q_1 - q_0$$  

(34)

where

$$q_0 = \begin{bmatrix} \gamma_0 \theta_0 \end{bmatrix}$$  

(35)

be the equilibrium value of the reduced state vector $q_1$. After the Mean Value Theorem, the closed-loop dynamics in (32) is expressed as

$$M_1 e_1 + V_{e_1} e_1 + K_1 e_1 + B_1 e_1 = w$$  

(36)

where

$$K_1 = \hat{K}_{1,2,1,2}.$$  

Consider the energy function

$$V(e_1, e_1) = \frac{1}{2} e_1^T M_1 e_1 + \frac{1}{2} e_1^T e_1.$$  

(37)

Taking time derivative of (37) and using the skew symmetric property in (6) yields

$$\dot{V} = e_1^T w - e_1^T B_1 \dot{e}_1 - e_1^T K_1 e_1 + e_1^T e_1.$$  

(38)

Adding and subtracting $\gamma^2 \|w\|^2 - \|z\|^2$ yields

$$\dot{V} = \gamma^2 \|w\|^2 - \|z\|^2 - \gamma^2 \|w\|^2 + e_1^T w - e_1^T B_1 \dot{e}_1 - e_1^T K_1 e_1 + e_1^T e_1$$

$$= \gamma^2 \|w\|^2 - \|z\|^2 - \gamma^2 \left( w - \frac{1}{2\gamma} \dot{e}_1 \right) + \frac{1}{4\gamma^2} \|\dot{e}_1\|^2$$

$$- e_1^T B_1 \dot{e}_1 - e_1^T K_1 e_1 + e_1^T e_1 + \phi_1 \|\dot{e}_1\|^2 + \phi_2 \|\dot{e}_1\|^2$$

$$\leq \gamma^2 \|w\|^2 - \|z\|^2 + \chi_1^T H_1 \chi_1.$$  

(39)

where

$$\chi_1 = \begin{bmatrix} e_1 \\ \dot{e}_1 \end{bmatrix}.$$  

(40)
Thus, following the assumption that the system is $\gamma$-dissipative yields the sufficient condition

$$\chi^T H_1 \chi \leq 0.$$  \hspace{1cm} (42)

which implies that $H_1$ is negative semi-definite throughout the entire trajectory of the system. Thus the worst-case gain$^1$ of the system is defined as

$$\gamma^* = \max_{\gamma} \left\{ \gamma \mid \chi^T H_1 \chi \leq 0, \quad \forall \chi \in \mathbb{R}^4 \right\}.$$  \hspace{1cm} (43)

**Theorem 1** Given a negative semi-definite matrix $H_1(\gamma)$ of the form given in (41), the worst-case gain defined on $H_1$ as

$$\gamma^* = \max_{\gamma} \left\{ \gamma \mid \chi^T H_1 \chi \leq 0, \quad \forall \chi \in \mathbb{R}^4 \right\}.$$  \hspace{1cm} (44)

is given by

$$\gamma^* = \frac{0.5}{\sqrt{Re\{\overline{\lambda}\} - \phi_1}}.$$  \hspace{1cm} (45)

where $\overline{\lambda}$ is the largest eigenvalue of $B_1 - \frac{K_1}{\phi_2}$.

**Proof.** The negative semi-definiteness of $H_1$ implies that $\text{tr}\{H_1\} \leq 0$ and $\text{det}\{H_1\} \geq 0$, which translates to the following conditions on the stiffness and damping matrices $K_1$ and $B_1$: \hspace{1cm} (54)

$$\text{tr}\{B_1\} \geq 2(\phi_1 + \phi_2) + \frac{1}{2\gamma^2}.$$  \hspace{1cm} (46)

$$\text{det}\left[ \frac{1}{4\gamma^2} + \phi_1 \right] I - \left( B_1 - \frac{K_1}{\phi_2} \right) \geq 0.$$  \hspace{1cm} (47)

Let

$$p(\lambda) = \text{det}\left[ \lambda I - \left( B_1 - \frac{K_1}{\phi_2} \right) \right]$$  \hspace{1cm} (48)

be the characteristic polynomial of the matrix $\left( B_1 - \frac{K_1}{\phi_2} \right)$, then the condition in (47) becomes

$$p\left( \frac{1}{4\gamma^2} + \phi_1 \right) \geq 0.$$  \hspace{1cm} (49)

Now, let $\lambda \leq \overline{\lambda}$ be the eigenvalues of $\left( B_1 - \frac{K_1}{\phi_2} \right)$, which correspond to the roots of $p(\lambda)$. The eigenvalues have positive real parts since the undisturbed system is assumed to have an asymptotically stable equilibrium. Thus, the condition in (49) is equivalent to the following conditions:

$$Re\{\overline{\lambda}\} \leq \frac{1}{4\gamma^2} + \phi_1.$$  \hspace{1cm} (50)

Putting (46), (50), and (51) together, the negative semi-definiteness of $H_1$ yields the following mutually exclusive conditions:

\begin{align*}
\text{either} & \quad \gamma^2 \geq \frac{0.25}{\text{max}\left[ \text{tr}\{B_1\} - \phi_1 - \phi_2, \left( Re\{\overline{\lambda}\} - \phi_1 \right) \right]} \\
\text{or} & \quad \frac{0.25}{\text{tr}\left( B_1 \right) - \phi_1 - \phi_2} \leq \gamma^2 \leq \frac{0.25}{\text{Re}\{\overline{\lambda}\} - \phi_1}. \hspace{1cm} (53)
\end{align*}

Thus, from (44), either $\gamma^* = \infty$ or $\gamma^*$ is given by (45). Since the system is stable, it cannot have an infinite gain. Therefore, the result in (45) is the worst-case gain.

**Remark 1** The stiffness and damping matrices $K_1$, and $B_1$ contain bounded functions of state and uncertain dynamic parameters which range between bounded values. Thus the worst-case gain of the system with constant stiffness can be lower bounded as

$$\gamma_1^* \geq \frac{0.5}{\sqrt{\lambda_1^* - \phi_1}} \equiv \gamma_1,$$  \hspace{1cm} (54)

where $\lambda_1^*$ is the smallest positive number larger than the largest singular value of $\left( B_1 - \frac{K_1}{\phi_2} \right)$, and $\gamma_1$ is termed the robust worst-case gain of the system. It characterizes the smallest worst-case gain.

$^1$ This is similar to the $H_\infty$-norm of the system.
3.2 Case 2: Passive Variable Stiffness

Here, the control mass is allowed to move under the influence of a restoring spring and damper forces. There is no external force generator added to the system. Only mechanical elements like the spring and damper are used. As a result, the system response is purely passive. Let \( k_u \) and \( b_u \) be the spring constant and damping coefficient of the restoring spring and damper respectively, then the control force \( u \) is given by

\[
u = -b_u \ddot{d} - k_u (d - l_{0_d}), \tag{55}\]

and the resulting dynamics of the control mass is given by

\[
m_u \ddot{d} + b_u \dot{d} + k_u (d - l_{0_d}) + k_s (\rho_s - 1)(d - x \cos \theta) + \frac{b_s}{2} g_{d\theta} \dot{\theta} + b_s g_s \dot{d} = 0, \tag{56}\]

and the static equilibrium equation for the control mass is given by

\[
k_u (d_0 - l_{0_d}) + k_s (\rho_s - 1)(d_0 - x \cos \theta_0) = 0, \tag{57}\]

where \( d_0 \) is the equilibrium position of the control mass, and \( l_{0_d} \) is the free length of the restoring spring. Let \( e_d = d - d_0 \) (58)

be the displacement of the control mass from its equilibrium position. Substituting (58) into (56) and using the Mean Value Theorem yields

\[
m_u \dddot{e}_d + B_d \ddot{e} + K_d \dot{e} = 0, \tag{59}\]

where

\[
e = \begin{bmatrix} e_1 \\ e_d \end{bmatrix}, \tag{60}\]

\[
B_d = \begin{bmatrix} b_u \\ b_s \frac{g_{d\theta}}{2} \\ b_s g_s + b_u \end{bmatrix}, \tag{61}\]

\[
K_d = \begin{bmatrix} 0 \\ k_s (\rho_s - 1)(d - x \cos \theta) \\ k_u + k_s (\rho_s - 1)(d - x \cos \theta) \end{bmatrix}_{\theta = \theta_0, \dot{\theta} = 0} \tag{62}\]

Now, consider the energy function

\[
V_2(e, \dot{e}) = \frac{1}{2} \dot{e}_1^T M_1 \dot{e}_1 + \frac{1}{2} e_1^T e_1 + \frac{1}{2} m_d \dot{e}_d^2 + \frac{1}{2} e_d^2. \tag{63}\]

Taking the first time derivative of (63), and following a similar procedure as in Case 1 yields

\[
\dot{V}_2 \leq \gamma^2 \|w\|^2 - \|e\|^2 + \mathcal{L}_2^T H_2 \mathcal{L}_2, \tag{64}\]

where

\[
\mathcal{L}_2 = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \tag{65}\]

\[
H_2 = \begin{bmatrix} \phi_2 I - K - B + \left( \frac{1}{4 \gamma^2} + \phi_1 \right) I \end{bmatrix}, \tag{66}\]

where

\[
\tilde{K} = \begin{bmatrix} \tilde{K} \\ \tilde{K}^T \end{bmatrix}, \tag{67}\]

\[
\tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{68}\]

where

\[
\tilde{K} = -\frac{\partial K}{\partial q} \begin{bmatrix} \delta q_{\zeta_1} \\ \delta q_{\zeta_2} \end{bmatrix}, \tag{69}\]

**Remark 2** Following a similar argument as in Remark 1, the robust worst-case gain of the system with a passive variable stiffness is given by

\[
\gamma_2 = \frac{0.5}{\sqrt{\lambda_2^* - \phi_1}} \tag{70}\]

where \( \lambda_2^* \) is the smallest positive number larger than the largest singular value of \( \frac{\tilde{B} - \tilde{K}^T}{\phi_2} \). Also, the spring constant \( k_u \), and the damping coefficient \( b_u \) of the control mass restoring spring-damper system can be chosen such that \( \lambda_2^* > \lambda_1^* \), which implies that \( \gamma_2 < \gamma_1 \). The outcome in (70) shows that a better performance can be achieved just by letting the stiffness vary naturally using a spring-damper system. This claim is supported in latter section by simulation results. This is a very appealing result due to its practicability. No additional electronically controlled or force generating device is required, only mechanical elements like the spring and damper are used.
4 SIMULATION

Table 1: DYNAMIC PARAMETER VALUES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_s$</td>
<td>358.859 kg</td>
<td>$k_s$</td>
<td>25900 N/m</td>
</tr>
<tr>
<td>$m_u$</td>
<td>44.1168 kg</td>
<td>$k_u$</td>
<td>15000 N/m</td>
</tr>
<tr>
<td>$m_d$</td>
<td>10 kg</td>
<td>$k_d$</td>
<td>210000 N/m</td>
</tr>
<tr>
<td>$b_s$</td>
<td>1500 Ns/m</td>
<td>$l_c$</td>
<td>0.015 kgm$^2$</td>
</tr>
<tr>
<td>$b_u$</td>
<td>1500 Ns/m</td>
<td>$l_{0s}$</td>
<td>0.8 m</td>
</tr>
<tr>
<td>$b_t$</td>
<td>600 Ns/m</td>
<td>$l_{0d}$</td>
<td>0.3 m</td>
</tr>
<tr>
<td>$l_{0t}$</td>
<td>0.35 m</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: KINEMATIC PARAMETER VALUES

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_l$</td>
<td>0.475 m</td>
</tr>
<tr>
<td>$h_u$</td>
<td>0.0775 m</td>
</tr>
<tr>
<td>$H$</td>
<td>0.4 m</td>
</tr>
<tr>
<td>$x$</td>
<td>0.45 m</td>
</tr>
</tbody>
</table>

The kinematic and dynamics parameter values used for the simulation are given in Tables 1 and 2. Time domain and frequency domain simulations were considered. The sprung mass $m_s$, unsprung mass $m_u$, stiffness $k_s$ and damping coefficient $b_s$ values are given in the "Renault Mégane Coupé" model [28]. The kinematic parameters $l_D$ and $h_u$ are chosen by measuring the rear suspension of Toyota highlander SUV. The others are chosen by rule of thumb. This is because the suspension system considered in this paper is not in production.

4.1 Time Domain Simulation

In the time domain simulation, the vehicle traveling at a steady horizontal speed of 40 mph is subjected to a road bump of height 8 cm. The Car Body Acceleration, Suspension Deflection, and Tire Deflection responses are compared between the constant stiffness and the passive variable stiffness cases. For the constant stiffness case, the control mass was locked at three different locations: $d = 20cm$, $d = 13.39cm$ and $d = 30cm$. The value $d = 13.39cm$ is the equilibrium position of the control mass. Next, a simulation is performed for the passive case. The results are reported in Figures 4, 5 and 6 which are the Car Body Acceleration, Suspension Deflection, and Dynamic Tire Force responses, respectively. Figure 7 shows the position history of the control mass for both the passive variable stiffness case.
Table 3: RMS (above) AND PEAK VALUES (below) OF SIMULATION RESULTS: Car Body Acceleration (CBA), Suspension Deflection (SD), Tire Deflection (TD)

<table>
<thead>
<tr>
<th></th>
<th>$d = 20\text{ cm}$</th>
<th>$d = 13.39\text{ cm}$</th>
<th>$d = 30\text{ cm}$</th>
<th>Passive</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBA ($m^2/s^2$)</td>
<td>0.8513</td>
<td>0.8766</td>
<td>1.0366</td>
<td>0.2216</td>
</tr>
<tr>
<td></td>
<td>3.0426</td>
<td>3.1293</td>
<td>3.6617</td>
<td>0.8293</td>
</tr>
<tr>
<td>SD ($cm$)</td>
<td>1.35</td>
<td>1.43</td>
<td>1.56</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>4.69</td>
<td>4.78</td>
<td>6.55</td>
<td>1.78</td>
</tr>
<tr>
<td>TD ($mm$)</td>
<td>2.0</td>
<td>2.0</td>
<td>2.3</td>
<td>1.4</td>
</tr>
<tr>
<td></td>
<td>5.4</td>
<td>5.5</td>
<td>10.8</td>
<td>3.7</td>
</tr>
</tbody>
</table>

Table 3 shows the rms and peak values of the results. It is seen that the variable stiffness cases show a significant improvement over their constant stiffness counterpart. Over 70% improvement in rms values in the car body acceleration and suspension deflection. There is also approximately 30% reduction in wear and tear caused on the road due to tire deflection. However, this reduction in tire deflection can be undesirable in performance cars like as race cars in which road holding constitute a very important factor. To meet the handling requirements in performance cars, the horizontal strut free lengths $l_{0d}$ and stiffness $k_u$ can be adjusted accordingly.

4.2 Frequency Domain Simulation

For the frequency domain simulation, an approximate frequency response from the road disturbance input to the performance vector given in (10), is computed using the notion of variance gain [15, 17]. The approximate variance gain is given by

$$G(j\omega) = \sqrt{\frac{2\pi N/\omega}{\int_0^{2\pi N/\omega} \frac{z^2}{A^2\sin^2(\omega t)} dt}}, \quad (71)$$

where $z$ denotes the performance measure of interest which is taken to be car body acceleration, suspension deflection, and tire deflection. The closed loop system is excited by the sinusoid $r = A\sin(\omega t)$, $t \in [0, 2\pi N/\omega]$, where $N$ is an integer big enough to ensure that the system reaches a steady state. The corresponding output signals were recorded and the approximate variance gains were computed using (71). Figures 8, 9, and 10 show the variance gain plots for the car body acceleration, suspension deflection, and tire deflection respectively. The figures show that the variable stiffness suspension achieves better vibration isolation in the human sensitive frequency range [1](4-8Hz), and better handling beyond the tire hop frequency[9] (> 59Hz).

5 CONCLUSION

The design and analysis of a new passive variable stiffness suspension system is presented. Using a detailed $L_2$-gain analysis based on the concept of energy dissipation, it is shown that inclusion of a variable stiffness mechanism in the suspension design yields an improvement in the performance of the system in terms of ride comfort, suspension deflection, and road holding. The result of the analysis is supported by both

Figure 8: Variance Gain: Car Body Acceleration

Figure 9: Variance Gain: Suspension Deflection

Figure 10: Variance Gain: Tire Deflection
time domain and frequency domain simulations. Future efforts will focus on building a prototype and studying the agreement of the subsequent experimental results with the analysis, and simulation results. The effect of variable stiffness on roll dynamics will also be examined using a half-car model.

REFERENCES


ABSTRACT
Since 1960, the US Gross Domestic Product (GDP) has increased from $520.5 billion to $14.6 trillion in 2010 [1]. As the United States has grown more productive and as more goods are consumed, more waste is generated. Of this waste stream, plastic refuse is problematic for society and has experienced low recycling rates. For example, the US generated 31 million tons of plastic waste in 2010 and of that total, only 8% was recycled [2]. A cost benefit analysis of the existing plastic sortation and recycling methods is necessary to better understand and promote the increased recycling of these materials. The process of turning plastic waste into reusable material at an industrial rate has pushed recycling into the 21st century. Technological advances in the recycling field have increased the efficiency and productivity of sortation processes. By using a standardized approach, the various methods of plastic recycling technology can quantified and compared.

This paper focuses primarily on the techniques used in modern recycling for the sortation of consumer plastic wastes. Each technique will be examined on a cost versus efficiency basis, and represents the variation used by large scale operations. Five techniques will be researched that include electrostatic separation; “sink/swim” differential method [3]; surfactant coated plastics [4]; infrared optical scanning [5]; and ultrasonic scanning. The aim of this paper is to build a detailed guide for analyzing plastic recycling methods and to determine the most cost effective methods.

INTRODUCTION
Recycling rates have expanded 28% in the past 20 years[6]. With demand for lower cost plastics increasing, and increased environmental concerns, recycling has not only become a necessity, but a profitable business. In just 2 years, recycling rates of plastics have increased by 8% alone[6]. This is expected to be greatly influenced by President Obama’s Recycling Works program. The program establishes a 75% solid municipal waste recycling goal for 2015[7]. It also dictates that 30% of all waste in the United States will be recycled, both by the manufactures and the consumers. This sets the precedent for major growth in the recycling industry in the next several years. Along with possible subsidies from the federal government, the program is expected to generate 1.5 million new jobs in both recycling and post-manufacturing waste recovery [7]. The government involvement has amazing potential for a complete redesign of the recycling market in the United States. The problem with the situation is that modern technologies are rarely used in current waste processing because the machines require large amounts of initial capital. If federal/state government subsidized capital for these operations, what technologies would they use? And how would they know that those technologies are superior to current methods?

In order to correctly analyze the current technologies presented, a control variable needed to be set. Plastic waste was chosen as a standard as it’s frequently recycled and very common to nearly every locality in the U.S. This makes it an ideal material control to be used in proposed recycling plants that would be funded by the federal government. Plastic waste has grown exponentially in U.S. landfills and the ease of acquiring usable waste for large scale operations is a desirable feature of using plastic. Plastic is also very durable, as well as being highly recyclable material, making it the ideal control material.

ANALYSIS
Within this paper, the present technologies available are examined, and which methods are most profitable are established. Similar parameters for the technology analyzed were chosen. The first recycling technology is the process of electrostatic separation. This is a fairly new method for separating plastics and is underutilized in the United States’ markets. The next two methods are a direct comparison of a common technique, using relative density in a liquid to separate plastics. The variant for these two technologies is the presence of a surfactant. The last technologies discussed in the
paper are methods of scanning plastics in order to quickly sort them, using other machines, such as infrared and ultrasonic scanners. The technologies stated above are the scenarios that will be used for quick reference. Electrostatic separation will be designated as Scenario 1. “Sink/Swim” Differentials and Surfactant aided separation will be the second and third scenarios, respectively. Lastly, infrared scanning will be the fourth scenario, and ultrasonic will be the fifth scenario.

We will examine each scenario with 4 standards, then conclude as to which method is the most profitable in terms of cost, efficiency, carbon footprint, and amount of energy consumed. Of the standards created for this paper, cost and efficiency are the chief constituents of the conclusion, and where an optimization will be created.

TECHNOLOGIES

Electrostatic Separation; Scenario 1

Electrostatic separation is the first methodexamined. This technology was developed in the 1990’s as an example of an electrically based solution to sorting plastic. Through statically charging particles using friction, then exposing them to an electrostatic field, an electromotive force is induced. This scenario is ideal because of the ease of using an electrical mean that sorts in several easy steps.

Mechanics

A simple tribo-electric separator consists of six components: a feeder system, a blower, a cyclone shaped tunnel for the tribo-electric friction to occur, assorted containers for collecting sortedplastic bins, and two vertical-plate electrodes along with their accompanying DC power supply. Figure 1 displays an overview of the system.

![Diagram of Electrostatic Separation](image)

The tribo-electric separator is typically installed with a climate control system, which regulates temperature and humidity. In operation, the feed is distributed in a current of air provided by the blower and introduced into the cyclone through its tangential entry. The air current is used to accelerate the mixture into the cyclone and rub it against the inner lining. After a certain period of frictional charging time (named as rubbing time), the oppositely charged plastics fall down freely in the area between the electrodes. The particles are drawn to either the positive or negative electrode according to the polarity of the charge, and separated by falling in different collecting bins.

Cost

The cost for Scenario 3 is typically lower than the other scenarios presented based on the simplicity of the design. Essentially, a very low tech version of this can be built exceptionally cheap, and due to the nature of the mechanics, the machine is accurate as well. Based on our scaling, the initial cost of this device would be roughly $250,000. The simplicity of the system makes this model considerably inexpensive compared to the other scenarios. With continuing research on the subject, cost may further be minimalized in future operations [8].

“Sink/Swim” Differential Method: Scenario 2

The method of sink/swim sorting of plastics is one of the older procedures used for recycling. The principle is very simple; mix the plastics in a large container filled with a liquid of known density. After some time, the plastics will either float or sink based on the density. This has been used since the 70’s and is a well-documented technique. In recent years, the system has grown largely in scale, due to increasing demands and further technological advancement of the practice.

Mechanics

Although this method is very straightforward in concept, the mechanics could be considered convoluted. Ideally, all of the plastic variations involved have prominent differences in density. The larger the contrast, the more accurately the plastic is sorted. Unfortunately, waste plastic densities do not differ significantly, as displayed in Table 1.

![Table 1: Density ranges for differential method](image)

<table>
<thead>
<tr>
<th>Density Range (g/cm³)</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polyolefins</td>
<td></td>
</tr>
<tr>
<td>Polypropylene</td>
<td>0.916-0.925</td>
</tr>
<tr>
<td>Low-density polyethylene</td>
<td>0.936-0.955</td>
</tr>
<tr>
<td>High-density polyethylene</td>
<td>0.956-0.980</td>
</tr>
<tr>
<td>Non-olefins</td>
<td></td>
</tr>
<tr>
<td>Bulk polystyrene</td>
<td>1.050-1.220</td>
</tr>
<tr>
<td>Polyvinyl chloride</td>
<td>1.304-1.336</td>
</tr>
<tr>
<td>Polyethylene terephthalate</td>
<td>1.330-1.400</td>
</tr>
</tbody>
</table>
In recycling, there is a narrow window for the process to work, so often containers are pressurized. This allows for slight changes in pressure, resulting in a diminutive modification of the density. The ability to minutely alter the density of the fluid creates a large amount of regulation over the process. By having an exact measure of the density of the liquid in relation to the desired plastic being sorted, much higher results are achievable. A simple computer based algorithm can separate the waste plastics by changing the pressure in the tank. This also significantly reduces the likelihood of having to adjust liquids to isolate alternative plastics.

**Cost**

The scenario runs efficiently based on a large scale system, and like many other large operations, the capital required for this project is exceptionally large. Based on the scaling for this comparison, a model that requires $400,000 dollars was selected, although there are significantly larger operations in existence. This model was considered based on the optimization of capital versus tons of waste processed[10].

**Surfactant Based Separation: Scenario 3**

The use of surfactants to effectively sort plastics on a large scale is also relevant. This method works exactly the same as a “Sink/Swim” system. When using this technique, the materials to be separated are first treated with a surfactant and then suspended in water. Because of a reaction with the surfactant material, plastics that would normally sink in water are suspended in the mixture. Air is then introduced into the system via pump. The air bubbles adhere to some particles depending on their resin type, causing the particles to float to the surface. Materials that are not affected by the bubbles sink to the bottom. Collection systems at the top and bottom of the tanks can then collect the now isolated materials.

The first noticeable benefit is that no advanced technology is essential. Second, the chemicals used are common in chemical processing and do not pose any substantial environmental hazards. Third, froth-flotation can separate certain plastics, such as PET from PVC, which has instituted a crucial problem to the conventional sink-float separation establishments.

**Cost**

The cost of scenario 3 is slightly higher than scenario 3, as there is a need for a more sophisticated tank system. This is reflected in the expense of $500,000 in only initial cost. Overall, this technique is particularly elaborate, yet is also notably more precise [11].

**Near-Infrared Scanning: Scenario 4**

Infrared scanning of plastics is not new to the recycling community, and has become one of the most common methods of sorting plastics. It’s used to sort nearly every type of plastic, and it can operate at large volumes. Using infrared scanners, the plastic is examined, and then it removed from the feed. It operates at extremely high accuracy, varying anywhere from 97% to 99% efficiency.

**Mechanics**

Infrared scanning operates on the concept that plastics can be analyzed using near-infrared scans that examine both density spectrometry and resin color. It recognizes the density spectrometry by exposing the plastic sample to infrared light, then processing the returning wavelength. It also uses simple high speed cameras to look for resin color. After identifying the type of plastics, it is marked optically, then that plastic is then removed from the system. Typically this is done instantly using compressed air to push the plastic off the line. During operation, the line moves rapidly, using an advanced computer system to track the plastics until their eventual removal from the machine.

**Cost**

Of all the scenarios examined during this paper, the infrared scanning technique is by far the most utilized technology. There are several companies that use near-infrared scanning to sort plastics. The method is favored as the most profitable way to sort plastics. The average initial capital required to purchase this equipment is $250,000 dollars. Due to market variation, this analysis is an average cost based multiple manufactures within the range of their potential efficiencies [12].

**Ultrasound Scanning: Scenario 5**

Ultrasonic scanning of plastics for recycling purposes is a more recent practice to the marketplace. This technology sorts plastics using ultra-sonic waves in water to determine density of plastic samples. The plastics are then removed from the processing line based on the results. This is typically done using mechanical arms to grab plastics and is a highly technical method of sorting. Ultrasonic scanning uses some of the same routes of technology utilized by other scenarios. Unlike other technologies, however, it can accurately describe plastic densities in non-clear liquids, such as ferro-fluid, whereas many optical methods would be useless. The ultrasound scanning method also builds a 3D-image of the objects, something which no other technology has employed.

**Mechanics**

Mechanics for this scenario are based on sounds propagation. First, the waste plastic is ground up into small pieces, roughly 20 mm. They are then submerged into a water-filled processing line, roughly 100 mm deep. The plastics are scanned using multiple ultrasonic waves at various wavelengths, which produces an accurate image of the plastics. Using mechanical means, the plastic is removed from the line.

**Cost**

Unfortunately, for this comparison, large scale production models for sorting plastics using ultrasonic scanning is not currently being employed. Although there are multiple small scale labs conducting testing as to whether ultrasonic scanning are viable, the method is still experimental.
DATA

Scenario 1

A simple tribo-electric generator used for sorting waste plastics can run anywhere from 1.75 to 2 tons per hour. This is based on a laboratory version set up for experimentation. The accuracy for scenario 1 is about 95% while the major cause of sorting failure was inability to properly charge shredded plastic waste [8].

Scenario 2

For scenario 2, the limiting factor for separation of plastic was the need for multiple containers in which to sink plastics. However, due to the large scale at which this can be done, the practice still remains a highly economical way to sort plastics.

In fact, the system can sort 3-4 metric tons per hour, which translates to roughly 2.95 to 3.93 imperial tons per hour.[10] The accuracy for this scenario is 95%-99%. The largest cause of sorting failure is from human error in the initial process, typically from poor cleaning prior to being submerged.

Scenario 3

In Scenario 3, the data was taken from the United States Dept. of Energy’s Argonne Laboratory. In this lab, a pilot was built to examine the feasibility of such operation. Based on the findings of the study, the system could analyze 2 tons per hour, which could be run in a single line.

Accuracy is 95%-99% for this scenario. The largest cause of sorting failure is from human error in the initial process, typically from poor cleaning prior to being submerged, similar to Scenario 2. [8]

Scenario 4

Scenario 4 is widely considered to be the most economical solution to sorting waste plastics. There are 5+ major manufacturers in the global market, and with that much capital involved in infrared sorting, the method has been widely researched, and is well documented. As per that basis, Scenario 4 can process roughly 2-4 tons per hour (Eveready Manufacturing). The accuracy for this scenario is 97%-99%, the largest cause of failure is from uneven stacking of the plastics in the conveyer system, preventing some plastics from being scanned [14].

Scenario 5

Scenario 5 has only been conducted in small scales and does not have a valid processing rate. Although it has a positive efficiency of 90% or greater based on small scale testing, it is still experimental [12].

CONCLUSIONS

Efficiency

A comparison of the five scenarios is displayed in Table 2. After reviewing all available data, Scenario 2 and 4 is the most desirable, producing the just less than 4 tons per hour. And their use is valid in large scale operations. Although they both operate in completely different ways, they are highly both highly accurate. Scenario 2 operates on the plastics' natural density to sink or swim when submerged into a liquid. Scenario 4 operates on the optical qualities of the plastic resins, therefore can be sorted quickly using scanning and a compressed air burst.

### Table 2. Method Comparison

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Equipment and installation cost</th>
<th>Processing capacity at system cost (tons per hour)</th>
<th>Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>$250,000</td>
<td>1.75-2.00</td>
<td>95%-98%</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>$4000,000</td>
<td>2.95 to 3.93</td>
<td>95%-99%</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>$500,000</td>
<td>2.00</td>
<td>95%-99.5%</td>
</tr>
<tr>
<td>Scenario 4</td>
<td>$250,000</td>
<td>2.00-4.00</td>
<td>97%-99%</td>
</tr>
<tr>
<td>Scenario 5</td>
<td>&lt;$100,000</td>
<td>Less than 1</td>
<td>90+%</td>
</tr>
</tbody>
</table>

Cost

The initial cost of scenario 1 and 4 clearly is the best solution based on cost at just $250,000. These costs were taken from market prices, and are estimates based on examples in the marketplaces. The difference in these are significant, Scenario 1 must be operated by a stand-alone machine, whereas scenario 4 is most likely built into a sorting line. It should also be noted that the cost of Scenario 1, if researched and backed by significant capital, initial cost should be greatly reduced. Based on the cost analysis for this comparison, scenario 1 & 4 are both valid.

SUMMARY

In conclusion, after significant research into the subject, the best solution for the large scale sortation of plastics is Scenario 4, infrared scanning. This scenario clearly dominates the marketplace currently, and for good reason, its low initial capital is certainly affordable and its processing rate is at the market forefront. It is also extremely low cost to operate; this is largely based on the small amount of labor needed to maintain the machine. Scenario 4 can process up to four tons per hour when operating at full capacity. At this rate the accuracy of sortation decreases, but still remains above the 95th percentile. Overall, as per our cost analytics, Scenario 4 or Near-Infrared Scanning is Best Solution to Processing and Sorting Waste Plastics.
REFERENCES
