# ON LEIGHTON'S COMPARISON THEOREM

Rudi Weikard

University of Alabama at Birmingham

Oberseminar Mathematische Physik
Universität München

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I am reporting on joint work with

- Ahmed Ghatasheh (UAB)

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  - $a$ and $b$ are consecutive zeros of $\tilde{u}$
- Then $u$ has a zero in $(a, b)$.
- Proof:

$$0 < \int_a^b (\tilde{q} - q)\tilde{u}u = \int_a^b (up\tilde{u}' - pu'\tilde{u})' = up\tilde{u}' \Big|_a^b \leq 0.$$

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  - $a$ and $b$ are consecutive zeros of $\tilde{u}$
- Then $u$ has a zero in $(a, b)$.
- Key to the proof is Picone's identity

$$\left(\frac{\tilde{u}}{u}(\tilde{p}\tilde{u}'u - \tilde{u}pu')\right)' = (\tilde{p} - p)\tilde{u}'^2 + (\tilde{q} - q)\tilde{u}^2 + \frac{p}{u^2}(\tilde{u}u' - \tilde{u}'u)^2$$

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- Leighton's integral is $\int_0^\pi [(p - \tilde{p})\tilde{u}'^2 + (q - \tilde{q})\tilde{u}^2] = \frac{\pi}{2}(k - \frac{\pi}{2})$
- If $k < \frac{\pi}{2}$, then $u$ has a zero in $(0, \pi)$.
- If $k \geq \pi/2$ there is again no conclusion.

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- Our goal was to develop comparison theorem for equations of this kind.

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- Comparison

# Relax continuity requirements, Part I

## Lemma

*If* $p > 0$, $\phi$ *absolutely continuous,* $\phi(a) = \phi(b) = 0$, *and*

$$\int_a^b (p\phi'^2 + q\phi^2) < 0,$$

*then every solution* $\psi$ *of* $-(pu')' + qu = 0$ *has a zero in* $(a, b)$.

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Proof:

- A variant of Picone's identity: if $\psi > 0$ on $(a, b)$ and $g = p\psi'\phi^2/\psi$, then

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- If $p, q, \phi'$ are continuous, $g$ vanishes at $a$ and $b$.
- Hence
$$0 \le \int_a^b p\psi^2(\phi/\psi)'^2 = \int_a^b (p\phi'^2 + q\phi^2) < 0$$

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- Cauchy-Schwarz: $\phi(x)^2 \leq k(x) \int_a^x p\phi'^2$. Thus

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- Recall $g = p\psi'\phi^2/\psi$.

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## Lemma

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*then every solution $\psi$ of $-(p(u' + su))' + rp(u' + su) + qu = 0$ has a zero in $(a, b)$. (Here $S$ and $R$ are antiderivatives of $s$ and $r$ which vanish at $a$.)*

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- and
$$-(p_0\psi_0')' + q_0\psi_0 = [-(p(\psi' + s\psi))' + rp(\psi' + s\psi) + q\psi]\mathrm{e}^{-R}.$$

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- Add (1) and (2).

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- If
$$\int_a^b [A(\tilde{u}' + \tilde{s}\tilde{u})^2 + B(\tilde{u}' + \tilde{s}\tilde{u})\tilde{u} + C\tilde{u}^2] \leq 0,$$
  where
  - $A = p\mathrm{e}^{2F+S-R} - \tilde{p}\mathrm{e}^G$
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# Special cases and examples

- Leighton's example revisited
- The generalized Sturm-Picone theorem
- The generalized Sturm separation theorem
- Jacobi difference equations
- The Schrödinger equation with a distributional potential

# Leighton's example revisited

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- Leighton's choice is $G = 0$ so that $C(x) = k - x$ and $\mathcal{I} = \int_a^b C\tilde{u}^2 = \frac{\pi}{2}(k - \frac{\pi}{2})$ but no conclusion for $k \geq \pi/2 \approx 1.571$.

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- Our choice of $G = 0.6x$ gives $\mathcal{I} < 0$ for $k \leq 1.672$. Therefore $u$ has then a zero in $(0, \pi)$.

# Leighton's example revisited

- $2F = G$ gives $A = B = 0$ and $C = C(x) = (k - x + g(x)^2/4)\mathrm{e}^{G(x)}$.
- Leighton's choice is $G = 0$ so that $C(x) = k - x$ and $\mathcal{I} = \int_a^b C\tilde{u}^2 = \frac{\pi}{2}(k - \frac{\pi}{2})$ but no conclusion for $k \geq \pi/2 \approx 1.571$.
- Our choice of $G = 0.6x$ gives $\mathcal{I} < 0$ for $k \leq 1.672$. Therefore $u$ has then a zero in $(0, \pi)$.
- Note that for $k \geq 1.676$ one can find solutions without zeros in $(0, \pi)$.

# Generalized Sturm-Picone theorem

- If
  - $Lu = 0$ and $\tilde{L}\tilde{u} = 0$
  - $\tilde{p} \geq p > 0$, $\tilde{q} > q$
  - $\mu = \tilde{p}(s - \tilde{s})\mathrm{e}^{-2S}$ non-decreasing on $[a, b]$
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- For the proof choose $r = s$, $\tilde{r} = \tilde{s}$, and $G = 2F = 2\tilde{S} - 2S$. Then $A \leq 0$, $C \leq 0$, and

$$\int_a^b B(\tilde{u}' + \tilde{s}\tilde{u})\tilde{u} = -\int_{[a,b)} \tilde{u}^2 \mathrm{e}^{2\tilde{S}} d\mu \leq 0.$$

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- $r = s$ was used only for simplicity. Also the condition that $\mu$ be finite near $a$ and $b$ can be relaxed.

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- Here $\tilde{p} = p$, etc.
- For the proof choose $F = 0$, $G = S - R$. Then $A = B = C = 0$.

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- $\alpha_{n-1}u_{n-1} + \beta_n u_n + \alpha_n u_{n+1} = 0$ where $N_0 + 1 \leq n \leq N_1 - 1$, $\alpha_n > 0$, $\beta_n \in \mathbb{R}$

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- Do not look for zeros but for sign changes.
- This situation is covered by our theorem:
  - $p = \alpha_n$ on $[n, n+1)$
  - $r = s = -\sum_{k=N_0+1}^{n} v_k / \alpha_n$ on $[n, n+1)$
  - $q = ps^2$
  - $u$ interpolates the values $u_n$ linearly.

# A comparison theorem for difference equations

- Suppose $-\tilde{\alpha}_n(\tilde{u}_{n+1} - \tilde{u}_n) + \tilde{\alpha}_{n-1}(\tilde{u}_n - \tilde{u}_{n-1}) + \tilde{v}_n \tilde{u}_n = 0$,

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- then $u$ changes sign on $[N_0, N_1]$.

# Distributions

- A linear functional $u$ on the test functions is called a distribution, if, for each compact set $K$, there are $C$ and $k$ such that

$$|u(\phi)| \leq C \sum_{j=0}^{k} \sup\{|\phi^{(j)}(x)| : x \in K \supset \operatorname{supp} \phi\}.$$

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- Every distribution has a derivative: $u'(\phi) = -u(\phi')$.
- Distributions also have anti-derivatives, any two differ by a constant: $u_1(\phi) - u_2(\phi) = C \int \phi$ if $u_1' = u_2'$ (Du Bois-Reymond)

# Sobolev spaces

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- If $v \in W^{-1,2}(a,b)$ and $u \in W^{1,2}(a,b)$, then $(vu)(\phi) = v(u\phi) = -\int V(u\phi)'$, so $uv \in W^{-1,2}(a,b)$.

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- On may now pose the equation

$$-u'' + vu = 0$$

whenever $v \in W^{-1,2}(a,b)$.

# The Schrödinger equation with a distributional potential

- $u$ is a solution of $-u'' + vu = 0$ then

$$0 = (-u'' + vu)(\phi) = \int u'\phi' - \int V(u\phi)' = \int (u' - Vu + W)\phi'$$

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- $p = 1$, $r = s = -V$, $q = -V^2$.

# Sturm's theorem for Schrödinger equations with a distributional potentials

- If
  - $v, \tilde{v} \in W^{-1,2}((a,b))$
  - $-u'' + vu = 0$ and $-\tilde{u}'' + \tilde{v}\tilde{u} = 0$
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- For the proof choose $F = G = 0$ and note that $\mu = \tilde{V} - V$ is non-decreasing.
- Then $A = B\tilde{s} + C = 0$ and

$$\int_a^b 2\mu\tilde{u}'\tilde{u} = \int_a^b \mu(\tilde{u}^2)' = -\int_{[a,b)} \tilde{u}^2 d\mu \leq 0.$$

# Vielen Dank für Ihre Aufmerksamkeit!