# The University of Alabama System

## Joint Ph.D Program in Applied Mathematics

## Linear Algebra and Numerical Linear Algebra JP Exam

## May 2025

## Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let $V$ be a vector space, and $T \in L(V)$ be a linear operator such that $T \circ T = T$. Prove that $\operatorname{Ker} T = \operatorname{Im}(I - T)$ and $V = \operatorname{Ker} T \oplus \operatorname{Im} T$.

2. (a) Let $A$ be an isometry on a finite-dimensional real inner product space $V$ which satisfies $A^2 = -I$. Prove that for every vector $\mathbf{v}$ in $V$, $A\mathbf{v}$ is orthogonal to $\mathbf{v}$.

   (b) Suppose $A$ in part (a) is an $n \times n$ real matrix. Find its eigenvalues and identify corresponding algebraic and geometric multiplicities. Is it possible to make a conclusion on $n$ as an odd or even number?

3. Let $V$ be a finite dimensional vector space over the complex field $\mathbb{C}$. For any linear operator $T \in L(V)$ and its eigenvalue $\lambda \in \mathbb{C}$, let $G(\lambda, T)$ denote the generalized eigenspace of $T$ corresponding to $\lambda$. Suppose $T$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for $\lambda \neq 0$.

4. Let $U$ and $W$ be subspaces of the finite dimensional inner product space $V$.

   (a) Show that $U^\perp \cap W^\perp = (U + W)^\perp$.

   (b) Show that $\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp)$.

5. Let $A \in \mathbb{R}^{m \times n}$ with $m \leq n$.

   (a) (4 points) Prove that $A$ is full rank if and only if $AA^T$ is invertible.

   (b) (6 points) Let $A$ now be of full rank. Prove that the matrix

   $$P = I - A^T(AA^T)^{-1}A$$

   is the orthogonal projection matrix of $\mathbb{R}^n$ onto $null(A)$.

6. Let $A \in \mathbb{R}^{n \times n}$ be strictly column diagonally dominant, i.e.,

   $$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|, \quad j = 1, \ldots, n.$$

   Show that $A$ has LU decomposition (without partial pivoting) and $A$ is nonsingular.

7. The Frobenius matrix norm of a matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

(a) Show that

$$\|A\|_F = \sqrt{tr(A^T A)} = \sqrt{tr(AA^T)},$$

where $tr(B)$ denotes the trace of $B$, the sum of its diagonal entries.

(b) Show that if $Q \in \mathbb{R}^{m \times m}$ is orthogonal, then $\|QA\|_F = \|A\|_F$. Then show that

$$\|A\|_F = (\sigma_1^2 + \cdots \sigma_r^2)^{1/2},$$

where $\sigma_i$ are nonzero singular values of $A$ for $r \leq \min(n, m)$.

8. (a) If $H \in \mathbb{C}^{n \times n}$ is a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, show that all eigenvalues of $H$ are real. Then show that for nonsingular $H$,

$$\kappa_2(H) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|},$$

where $\kappa_2(H)$ is the 2−norm condition number.

(b) Suppose $A \in \mathbb{R}^{n \times n}$ has a singular value decomposition $A = U \Sigma V^T$, where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, $U = [u_1, \ldots, u_n]$ and $V = [v_1, \ldots, v_n]$, with $u_i, v_i \in \mathbb{R}^n$. Find the eigenvalues and corresponding eigenvectors of the matrix

$$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

Your results need to be in terms of $\sigma_i, u_i$ and $v_i$, $i = 1, \ldots, n$.