Joint Program Exam in Mathematical Analysis
August 11, 2025

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two “half solutions” to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.

1. Let (X, d) be a metric space, and $K \subset X$ be a compact subset. Let $\{U_n\}_n$ be a sequence of open sets, which cover K, i.e. $K \subset \cup_{n=1}^{\infty} U_n$. Prove that there exists $\epsilon_0 > 0$ such that for every $x \in K$, there is n, so that $B(x, \epsilon_0) \subset U_n$.

2. Let $\{a_n\}_{n=1}^{\infty} : a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} > 0, n = 1, 2, \ldots$. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

converges.

3. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = \frac{j}{2^n}, j \text{ are odd integers}, 0 < j < 2^n, n = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Determine if f is Riemann integrable on $[0, 1]$ and justify your answer.

4. Let $f_n : \mathbb{R} \to \mathbb{R}, n = 1, 2, \ldots$ be differentiable functions, with $|f_n'(x)| \leq L$. Suppose that $f_n \to f$ pointwise, that is $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in \mathbb{R}$. Prove that

a) f is continuous on $\mathbb{R}$;

b) the convergence $f_n \to f$ is uniform on the compact subsets of $\mathbb{R}$.

Hint: For part *b*), it suffices to show that $f_n \to f$ uniformly on $[-M, M]$ for each $M > 0$.

5. Prove that $f(x) = \sum_{n=1}^{\infty} n^2 e^{-nx}$ is continuously differentiable over $(0, \infty)$. Then prove that $\lim_{x\to 0+} f(x) = +\infty$.

6. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable and also convex function, i.e. for all $a < x < y < b$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}.$$

Prove that

$$\int_a^b f(x)dx \geq (b - a) f\left(\frac{a+b}{2}\right).$$

7. Suppose that $f : (0, \infty) \to \mathbb{R}$ is twice differentiable and that both f and f'' are bounded on $(0, \infty)$.

a) Prove that f' is also bounded on $(0, \infty)$.

b) Suppose further that $\lim_{x\to\infty} f(x)$ exists (as a finite number). Prove that $\lim_{x\to\infty} f'(x) = 0$.