The University of Alabama at Birmingham

Department of Mathematics

MA 125.5c | Calculus I | DRS/Autumn/2003

Final Examination, Part I

Do Not Write Your Name On This Sheet

Do not write your name on the front side of any page.
Do write your name on the back side of the last page.

Read these instructions: In the space provided, include a neatly written argument leading to the solution. Be sure to include appropriate details. *Read these problems very carefully. Make certain that you have fully responded.* You may use the back side of any sheet for scratch work. If you need additional scratch paper, I will provide it upon request.

You must show your work!

1. Compute each of the following limits. Try to not resort to the use of l'Hospital's rule. Problems in which l'Hospital's rule is used will earn only half-credit. If the limit fails to exist, so state.

   (a) $\displaystyle\lim_{x\to 2} \frac{x^2+3x-10}{x^2-4}$

   (b) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{5x}$

2. (a) Complete the following statement: The statement that the function function f is continuous at the number a means that ____

   ____

   (b) Fill in the blank so that the function is continuous everywhere. If the function cannot be extended to a continuous function, then in the space to the right of the problem, so state and explain why it cannot be extended.

   i. $f(x) = \begin{cases} \sin\frac{1}{x}, & \text{if } x \neq 0 \\ \_\_\_\_, & \text{if } x = 0 \end{cases}$

   ii. $f(x) = \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq 0 \\ \_\_\_\_, & \text{if } x = 0 \end{cases}$

   iii. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ \_\_\_\_, & \text{if } x = 0 \end{cases}$

3. (a) Complete the following definition: The statement that f' is derived from the function f means that ____

   ____

   (b) Using the above definition, find the function derived from $f(x) = \sqrt{x}$.

4. (a) After proving the chain rule, I summarized it by a certain statement. Fill in the blanks in order to reproduce that statement.

   If $y = f(z)$ and $z = g(x)$, then $y = F(x) =$ ________ and $F'(x) =$ ______________.

(b) You may assume that $\mathbf{D} \sin x = \cos x$. Use this fact and the chain rule in order to prove that $\mathbf{D} \sin g(x) = \cos g(x)\, g'(x)$.

5. (a) Early in the term we graphed functions of the form $y = a^x$ for several different values of a. There was only one of these values that we named e. What property did the graph of $y = e^x$ have that no other graph of the form of $y = a^x$ have?

   (b) Prove the product rule.

6. Perform the indicated differentiation or antidifferentiation.

   (a) $\mathcal{D}\, x^2 e^{x^3}$

   (b) $\mathcal{A}\, x^2 e^{x^3}$

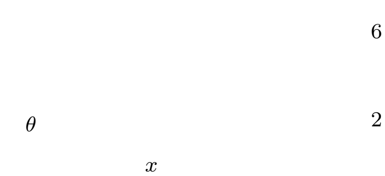
7. (a) $\mathcal{D}\ (\sin x + \cos x)^2$

   (b) $\mathcal{A}\ (\sin x + \cos x)^2$

8. (a) Find the slope of the graph of $y + \ln(xy) = 1$ at the point $(1,\, 1)$.

   (b) Sketch the following two graphs. No discussion is required, so draw your graph carefully.

9. (a) A picture whose height is six feet is hanging so that the bottom edge is two feet above eye level. How far back should an observer stand in order to maximize his view of the picture?



   (b) $\mathcal{A}\, x^5 e^{x^3}$

   Hint: As you may have noticed, while we rewrote many of our differentiation forms into their corresponding antidifferentiation forms, we did not do so with the product rule. We now rectify this apparent oversight.

$$\mathcal{D}\,(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

   may be rewritten as

$$\begin{aligned} \mathcal{A}\big(f(x)g'(x) + g(x)f'(x)\big) &= f(x)g(x) \\ \mathcal{A}\, f(x)g'(x) + \mathcal{A}\, g(x)f'(x) &= f(x)g(x) \\ \mathcal{A}\, f(x)g'(x) &= f(x)g(x) - \mathcal{A}\, g(x)f'(x) \end{aligned}$$

   This form of the product rule enables us to exchange one antiderivative for another. If we can do so in such a way as to replace a complicated antiderivative by simpler one, then we have made progress. For example:

$$\begin{aligned}\mathcal{A}\, x\cos x &= x\sin x - \mathcal{A}\,\sin x\\ &= x\sin x + \cos x\end{aligned}$$

As you can see, I have used the above form of the product rule with $f(x) = x$ and $g'(x) = \cos x$. Finally, I will check my work by differentiation:

$$\begin{aligned}\mathcal{D}\,(x\sin x + \cos x) &= \mathcal{D}\, x\sin x + \mathcal{D}\,\cos x\\ &= x\cdot\cos x + \sin x\cdot 1 - \sin x = x\cos x.\end{aligned}$$

Second Part

1. (a) Sketch the graph of $y = \ln(1 + x^2)$. Find each relative maximum, relative minimum and point of inflection.
   (b) Suppose the function $f(x) = x^4 - 8x^2 + 6$ is defined over the closed interval $[-1,\, 3]$. Find the maximum and minimum value of this function and state where each occurs.

2. You are to design a cylindrical tin can that will contain 80 cubic inches. However, you can purchase the materials only in the form of rectangles. There is no waste in the rectangle used for the cylinder; however, the corners of the squares from which the ends are cut are wasted. Nonetheless they constitute part of your cost. The material used for the top of the can cost 2 cents per square inch and the material used for the bottom and the side-wall cost 1 cent per square inch. Find the dimensions that will minimize your cost of materials.

3. You are designing a poster to contain 50 square inches of printing with margins of 4 inches each at top and bottom and 2 inches at each side. You are to write down an expression for the function that you need to optimize in order to minimize the amount of paper used. Write this expression as a function of a single variable and state whether you need to find a maximum or a minimum of the function.

4. (a) i. Write down a statement of Rolle's Theorem.
   ii. Write down a statement of the Extended Mean Value Theorem
   (b) Prove that the Extended Mean Value Theorem is a consequence of Rolle's Theorem.

5. (a) Find the area of the region bounded by the graph of $y = 9 - x^2$ and the x-axis.
   (b) Suppose that $f(x) = x^3 - 2x^2$ and $g(x) = x - 2$. Find the area of the region bounded by the graphs of f and g. Express your answer in terms of symbols of the form $\mathbf{A}_a^b h(x)$, where h, a and b are appropriately defined.

6. During the construction of the Hoover Dam on the border of Arizona with Nevada, a worker standing on a ledge 400 feet above the canyon floor happened to drop a hammer. With what speed did it strike the ground 400 feet below? You may assume that the acceleration due to gravity is -32 feet/second/second.

7. (a) Prove that the function $f(x) = 2x$ has only one point in common with the function $g(x) = \cos x$
   (b) Write down Newton's recursion relation for determining the above point of intersection.

8. A certain snowball is melting in such a way that it maintains the shape of a sphere throughout the melting process. By collecting the melt, we are able to observe that at the moment when diameter is 10 centimeters, the volume is decreasing at the rate of 5 cubic centimeters per minute. Find the rate at which the surface area is decreasing at this moment.
   Hint: The surface area S of a sphere of radius r is $S = 4\pi r^2$ and the volume V of that same sphere is $V = \frac{4}{3}\pi r^3$.

9. (a) $\mathcal{D}\ \sin^x x$

   (b) $\mathcal{A}\ \sin^3 x$

10. (a) $\displaystyle \mathcal{A}\ \frac{x}{4x^2+1}$

   (b) $\displaystyle \mathcal{A}\ \frac{1}{4x^2+1}$