FALL 2009 — MA 227-6B — TEST 2
OCTOBER 14, 2009

Name: ______________________________

1. Part I

There are 6 problems in Part 1, each worth 4 points. Place your answer on the line to the right of the question. Only your answer on the answer line will be graded.

(1) Solve the equation $2x^2 + \pi x - 5 = 0$ for x. $\underline{-\frac{\pi}{4} \pm \frac{1}{4}\sqrt{\pi^2 + 40}}$

(2) Lance's position is described by the vector function $\vec{r}(t) = \langle 2 + 3t^2, 3 + 4t^2 \rangle$ during the time interval $[0, 5]$. Find his speed when $t = 3$.

$\underline{30}$

(3) Describe or sketch the domain of the function $f(x, y) = \sqrt{8 - 2x^2 - 2y^2}$.

<u>The disc of radius 2 centered at the origin including the boundary.</u>

(4) A bug crawls on a metal plate along curve given by $\vec{r}(t) = \langle 2t + 1, t^2 - 1 \rangle$. The temperature on the plate is given by $T(x, y, t) = 5 - 2xy + 3yt$. How warm is the spot the bug reaches at $t = 2$ at that time?

$\underline{-7}$

(5) Find the gradient of $f(x, y) = xy + \sqrt{y^2 - 2}$.

$\underline{\langle y, x + y/\sqrt{y^2 - 2} \rangle}$

(6) Find the tangent plane to the paraboloid $z = 5 - x^2 - y^2$ at the point $(0, 0, 5)$.

$\underline{z = 5}$

2. Part II

There are 2 problems in Part 2. The number of points to earn is indicated in each problem. On Part 2 problems partial credit is awarded where appropriate. Your solution must include enough detail to justify any conclusions you reach in answering the question. Points will be awarded for the correct reasoning. If you need to add assumptions in order to proceed, make the assumptions after explaining why they are reasonable.

(1) (24 points) Consider the function $f(x,y) = \sqrt{4x+y^2}$ for $x, y \geq 0$ and the line L given by $4x - 3y = 5$.
   (a) Check whether the points $P = (2,1)$ and $Q = (8,9)$ are on the line L.
   (b) What is the distance between P and Q?
   (c) Find a parametrization for L which has constant speed one. Which values of the parameter correspond to P and Q?
   (d) Find the profile function F of the function f along the line L? (Imagine a perpendicular cut through the graph of the function by the plane $4x - 3y = 5$.)
   (e) The values of F and f should be the same for corresponding points (well that's exactly the idea of the profile function). Check that this is so for P and Q.
   (f) Find the directional derivative of f at the point P in the direction towards the point Q using the profile function.
   (g) State a formula for directional derivatives using the gradient of f.
   (h) Compute the directional derivative using this formula.
   (i) Compare the two results.

   Solution:
   (a) For a point to be on the line its coordinates have to satisfy the equation. Since $4(2) - 3(1) = 5$ and $4(8) - 3(9) = 5$ both points are on the line.
   (b) The vector joining the points is $\langle 6, 8\rangle$. Its length is their distance, so the distance is $\sqrt{36+64} = 10$.
   (c) $\vec{r}(t) = \langle 2,1\rangle + \langle 6,8\rangle t$ passes through P and has the direction so it is a parametrization of the line but with the wrong speed (speed is 10). So reducing the speed by a factor of 10 gives the sought parametrization: $\vec{r}(t) = \langle 2,1\rangle + \langle 6,8\rangle t/10$.
   (d) $F(t) = f(\vec{r}(t)) = \sqrt{9 + 4t + 16t^2/25}$.
   (e) Point P equals $\vec{r}(0)$. We have $F(0) = 3 = f(2,1)$. Point Q equals $\vec{r}(10)$. $F(10) = \sqrt{9+40+64} = \sqrt{32+81} = f(8,9)$.
   (f) The directional derivative at P is, by definition, the derivative of the profile function at the point corresponding to P, that is $t = 0$. $F'(t) = (4 + 32t/25)/(2\sqrt{9+4t+16t^2/25})$. $F'(0) = 4/(2 \times 3) = 2/3$.
   (g) We showed in class that the directional derivative at P in direction of the unit vector $\vec{u}$ is given by $(\nabla f)(P) \cdot \vec{u}$.
   (h) $\vec{u} = \langle 6,8\rangle/10 = \langle 3,4\rangle/5$, $\nabla f = \langle 4, 2y\rangle/(2\sqrt{4x+y^2})$, and $(\nabla f)(P) = \langle 4,2\rangle/6 = \langle 2,1\rangle/3$ give $(D_{\vec{u}} f)(P) = \langle 3,4\rangle \cdot \langle 2,1\rangle/15 = (6+4)/15 = 2/3$.
   (i) The results in (f) and (h) coincide as it must be.

(2) (12 points) Consider the function $h(x, y) = \sqrt{xy}$.

(a) Sketch the contour lines (level curves) of h in the first quadrant.

(b) Let P be the point $P = (4, 9)$. Compute the value of h at P.

(c) Find the linear approximation $L(x, y)$ of h at P.

(d) Estimate the value of h at the point $(3.9, 9.1)$.

Solution:

(a) The level curves are the curves where h is constant, but if $\sqrt{xy} = c$ then $xy = c^2$. If $c = 0$ these are the coordinates axes. Otherwise $y = c^2/x$ (since x cannot be zero). These curves are hyperbolas.

(b) $h(P) = h(4, 9) = \sqrt{36} = 6$.

(c) The linear approximation to h at $P = (x_0, y_0)$ is

$$L(x, y) = h(P) + (\nabla h)(P) \cdot \langle x - x_0, y - y_0\rangle.$$

Here $\nabla h = \langle y, x\rangle/(2\sqrt{xy})$ and $(\nabla h)(P) = \langle 9, 4\rangle/12$ so that

$$L(x, y) = 6 + \frac{1}{12}\langle 9, 4\rangle \cdot \langle x - 4, y - 9\rangle = \frac{3x}{4} + \frac{y}{3}.$$

(d) $h(3.9, 9.1)$ is approximated by $L(3.9, 9.1)$ which can be computed explicitly:

$$L(3.9, 9.1) = 6 + \frac{1}{12}\langle 9, 4\rangle \cdot \langle -0.1, 0.1\rangle$$
$$= 6 + (-9 + 4)/120 = 6 - 5/120 = 6 - 1/24.$$

The point $(3.9, 9.1)$ is close to $(4, 9)$ and the function values are also close ($1/24 \approx 0.042$).