FALL 2009 — MA 227-6B — TEST 4
DECEMBER 2, 2009

Name: ______________________________

1. Part I

There are 6 problems in Part 1, each worth 4 points. Place your answer on the line to the right of the question. Only your answer on the answer line will be graded.

(1) Find the rectangular coordinates of the point whose spherical coordinates are given by $\rho = 3$, $\phi = \pi/3$, and $\theta = \pi/2$.

$\langle 0, 3\sqrt{3}/2, 3/2\rangle$

(2) For the point with rectangular coordinates $(3, -2, -6)$ find $\cos\phi$ (here (ρ, θ, ϕ) denote spherical coordinates).

$-6/7$

(3) Compute $\text{grad}\, f$ $(= \nabla f)$ when $f(x, y, z) = x + y^2 + z$.

$\langle 1, 2y, 1\rangle$

(4) Find a parametrization for the line joining the points $P = (2, 0)$ and $Q = (0, 3)$.

$\langle 2 - 2t, 3t\rangle$, $0 \leq t \leq 1$

(5) Find a function f such that $(\nabla f)(x, y) = \langle 2xy + 3, x^2\rangle$.

$x^2y + 3x$

(6) Find the line integral $\int_C 5ds$ when C is the quarter-circle $\langle \cos(t), \sin(t)\rangle$, $\pi/2 \leq t \leq \pi$. You may do this by computation or by reasoning.

$5\pi/2$

2. Part II

There are 3 problems in Part 2. Choose two of the problems to work on. Each of these two is worth 18 points. On problems in Part 2 partial credit is awarded where appropriate. Your solution must include enough detail to justify any conclusions you reach in answering the question. Points will be subtracted if the reasoning is missing or incorrect.

(1) Using spherical coordinates, evaluate the triple integral $\iiint_E z\,dV$, where E is that portion of the ball of radius 4 centered at zero for which $z \geq 0$.

Solution:

All points in the ball satisfy $0 \leq \rho \leq 4$. Since $z = \rho\cos\phi$ the requirement $z \geq 0$ is equivalent to $0 \leq \phi \leq \pi/2$. We have no restriction on θ, i.e., $0 \leq \theta \leq 2\pi$. Also $dV = \rho^2 \sin\phi d\rho d\phi d\theta$. Hence

$$
\begin{aligned}
\iiint_E z\,dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^4 \rho\cos\phi\rho^2\sin\phi d\rho d\phi d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/2} 2\sin\phi\cos\phi\int_0^4 \rho^3 d\rho d\phi d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/2} 2\sin\phi\cos\phi\frac{1}{4}\rho^4\Big|_{\rho=0}^{\rho=4} d\phi d\theta \\
&= \frac{4^3}{2}\int_0^{2\pi}\int_0^{\pi/2} \sin(2\phi) d\phi d\theta \\
&= \frac{4^3}{2}\int_0^{2\pi} -\frac{1}{2}\cos(2\phi)\Big|_{\phi=0}^{\phi=\pi/2} d\theta \\
&= \frac{4^3}{2}\int_0^{2\pi} d\theta \\
&= 4^3\pi = 64\pi.
\end{aligned}
$$

(2) Find the work done by the force field $\mathbf{F} = \langle 3 + 1/x, 1/y \rangle$ on a particle that moves along a line segment from the point $(1, 1)$ to the point $(2, 4)$.

First Solution:

Work is given by the line integral $\int_C \mathbf{F} \cdot d\,\mathbf{r}$. A parametrization of C is given by $\mathbf{r}(t) = \langle 1 + t, 1 + 3t \rangle$ where $0 \leq t \leq 1$. Therefore $\mathbf{r}'(t) = \langle 1, 3 \rangle$.

Since $\mathbf{F}(x, y) = \langle 3 + 1/x, 1/y \rangle$ we get $\mathbf{F}(\,\mathbf{r}(t)) = \langle 3 + 1/(1 + t), 1/(1 + 3t) \rangle$. Hence

$$\mathbf{F}(\,\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 3 + 1/(1 + t) + 3/(1 + 3t).$$

$\int_C \mathbf{F} \cdot d\,\mathbf{r} = \int_0^1 (3 + \frac{1}{1+t} + \frac{3}{1+3t})dt = 3 + \log(2) + \log(4) = 3 + 3\log(2).$

Second Solution:

F is conservative with potential $f(x, y) = 3x + \log(xy)$. The line integral equals then $f(2, 4) - f(1, 1) = 6 + \log(8) - (3 + \log(1)) = 3 + 3\log(2)$.

(3) Use Green's theorem to evaluate the line integral

$$\int_C [\left(2y^2 + \ln(1+\sqrt{x})^3\right) dx \, + \left(3xy + \arctan(y^2 + y^3 \mathrm{e}^y)\right) dy]$$

where C consists of the parabolic segment of $y = x^2$ joining the points $(-2, 4)$ and $(2, 4)$ and the line segment joining and these two points. Assume the counterclockwise orientation for C. Hint: The problem looks harder than it is.

Solution:

Set $P = 2y^2 + \ln(1+\sqrt{x})^3$ and $Q = 3xy + \arctan(y^2 + y^3\mathrm{e}^y)$. By Green's theorem the given line integral equals

$$\iint_D (Q_x - P_y) dA$$

where D is the domain between the two curves, i.e., $x^2 \le y \le 4$ and $-2 \le x \le 2$. Since $Q_x - P_y = -y$ we get for the value of the integral

$$\int_{-2}^{2}\int_{x^2}^{4} -y dy dx = -\frac{1}{2}\int_{-2}^{2} y^2\Big|_{x^2}^{4} dx = -\frac{1}{2}\int_{-2}^{2}(16 - x^4)dx = -\frac{1}{2}(16x - \frac{1}{5}x^5)\Big|_{-2}^{2} = -\frac{128}{5}.$$