NAME:______________________________________

GRADE:_____________________________________

SCHOOL NAME:__________________________

2023-2024 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (10 pts) In the formula $101 - 102 = 1$ move one digit so that it becomes a true identity.

YOUR ANSWER:

PROBLEM 2 (20 pts) There are 4 boxes in each of which there is a pen, a pencil, and a candy. These objects are of the following colors. In the first box there are purple pen, green pencil, and red candy. In the second box there are a blue pen, a green pencil, and a yellow candy. In the third box there are a purple pen, an orange pencil, and a yellow candy. Notice that so far for any two boxes there is exactly one similar object that they share (for example, the common object for the first and the second box is a green pencil). What should we put in the 4th box so that this condition is kept?

YOUR ANSWER:

PROBLEM 3 (40 pts) A car left Birmingham at 2pm and arrived in New Orleans at 7pm. Another car left New Orleans at 3pm and arrived in Birmingham at 8pm. The car went along the same route and moved with the same constant speed. At what time did they meet?

YOUR ANSWER:

over, please

PROBLEM 4 (80 pts) A quadratic polynomial $f(x) = ax^2 - ax + 1$ with real coefficients is such that $|f(x)| \leq 1$ for any $0 \leq x \leq 1$. Find the greatest possible value of a satisfying this condition.

YOUR ANSWER:

PROBLEM 5 (100 pts) Quadratic polynomials $f(x) = ax^2 + bx + c$ and $g(x) = ax^2 + dx + u$ with real coefficients have two real roots each. It is known that the sum of their four roots equals 2024. It is also known that $h(x) = f(x) + g(x)$ has two real roots. Find the sum of the roots of $h(x)$.

YOUR ANSWER:

PROBLEM 6 (130 pts) Given a real number x denote by $[[x]]$ the *integer part* of x, that it the greatest integer n such that $n \leq x$. For example, $[[.5]] = 0$ while $[[7]] = 7$. Find the sum

$$[[1/3]] + [[2/3]] + [[4/3]] + \cdots + [[2^{49}/3]] + [[2^{50}/3]].$$

Give an answer either in the numerical form (as a number) or in the form of an expression (not involving the integer part function).

YOUR ANSWER:

PROBLEM 7 (250 pts) Jack and Jill play the following game. Jack writes a sign (either + or -) on a large board. Then Jill writes a number from 1 through 2021 next to it, and so on. This is repeated 2021 times until the collection of numbers from 1 through 2021 is exhausted (Jill can use each number only once). If the resulting sum is x, then Jill gets $\$|x|$ award. What maximal award \$A can Jill **guarantee** to receive? The justification must include Jack's strategy preventing Jill from getting more than \$A and Jill's strategy guaranteeing that at the end she will get at least \$A.

YOUR ANSWER:

2023-2024 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) In the formula $101 - 102 = 1$ move one digit so that it becomes a true identity.

Solution: It is easy to see that

the answer is $\mathbf{101 - 10^2 = 1}$. □

PROBLEM 2 (20 pts) There are 4 boxes in each of which there is a pen, a pencil, and a candy. These objects are of the following colors. In the first box there are purple pen, green pencil, and red candy. In the second box there are a blue pen, a green pencil, and a yellow candy. In the third box there is a purple pen, an orange pencil, and a yellow candy. Notice that so far for any two boxes there is exactly one similar object that they share (for example, the common object for the first and the second box is a green pencil). What should we put in the 4th box so that this condition is kept?

Solution: It is straightforward to verify that

the answer is **blue pen, orange pencil, red candy**. □

PROBLEM 3 (40 pts) A car left Birmingham at 2pm and arrived in New Orleans at 7pm. Another car left New Orleans at 3pm and arrived in Birmingham at 8pm. The cars went along the same route and moved with the same constant speed. At what time did they meet?

Solution: Let us denote time when they met by t. By that time the first car traveled for $t - 2$ hours. After that time the second car traveled for $8 - t$ hours. In both cases the cars covered the same distance (the distance from Birmingham to the point where they met) and did it with the same speed. Therefore they needed the same time to do so. It follows that $t - 2 = 8 - t$ and hence that $t = 5$.

The answer is **5pm**. □

PROBLEM 4 (80 pts) A quadratic polynomial $f(x) = ax^2 - ax + 1$ with real coefficients is such that $|f(x)| \leq 1$ for any $0 \leq x \leq 1$. Find the greatest possible value of a satisfying this condition.

Solution: The graph of $f(x)$ must pass through points $(0,1)$ and $(1,1)$ on the coordinate plane. Moreover, it must also pass through the point $(\frac{1}{2}, 1-\frac{a}{4})$. Hence any number a satisfying the condition $|f(x)| \leq 1$ must be non-negative. It follows that the maximal value of $f(x)$ on $[0,1]$ is then 1 (assumed at 0 and 1) while the minimal value is $1-\frac{a}{4}$ (assumed at $\frac{1}{2}$). Hence the greatest value of a satisfying the condition $|f(x)| \leq 1$ is the number a solving the equation $1-\frac{a}{4} = -1$, that is the number $a = 8$.

The answer is $\mathbf{a = 8}$. □

PROBLEM 5 (100 pts) Quadratic polynomials $f(x) = ax^2 + bx + c$ and $g(x) = ax^2 + dx + u$ with real coefficients have two real roots each. It is known that the sum of their four roots equals 2024. It is also known that $h(x) = f(x) + g(x)$ has two real roots. Find the sum of the roots of $h(x)$.

Solution: If we divide $f(x)$ and $g(x)$ by a, then the roots of these polynomials will not change. Moreover, the sum of new $f(x)$ and $g(x)$, that is the new $h(x)$, will be the quotient of the old $h(x)$ and a which implies that the roots of $h(x)$ will not change either. Hence we may from the very beginning assume that $a = 1$. Denote the roots of $f(x)$ by t_1 and t_2. Denote the roots of $g(x)$ by v_1 and v_2. By Vieta's formula and under our assumption that $a = 1$ we see that $t_1 + t_2 = -b$ and $v_1 + v_2 = -d$. Since the sum of the roots of both equations is 2024, then $-b - d = 2024$. Now, by Vieta's formula the polynomial $h(x)$ has two roots with the sum equal to $\frac{-b-d}{2}$ which implies that the sum of its roots is 1012.

The answer is **1012**. □

PROBLEM 6 (130 pts) Given a real number x denote by $[[x]]$ the *integer part* of x, that it the greatest integer n such that $n \leq x$. For example, $[[.5]] = 0$ while $[[7]] = 7$. Find the sum

$$[[1/3]] + [[2/3]] + [[4/3]] + \cdots + [[2^{49}/3]] + [[2^{50}/3]].$$

Give an answer either in the numerical form (as a number) or in the form of an expression (not involving the integer part function).

Solution: The explicit formula for $[[2^n/3]]$ depends on the parity of n. If $n = 2m$ is even then 2^{2m} has the remainder 1 when divided by 3 which implies that in this case $[[2^n/3]] = \frac{2^n-1}{3} = \frac{2^n}{3} - \frac{1}{3}$. If $n = 2m + 1$

is odd then 2^{2m+1} has the remainder 2 when divided by 3 which implies that in this case $[[2^n/3]] = \frac{2^n-2}{3} = \frac{2^n}{3} - \frac{2}{3}$. Hence the sum considered in the problem consists of the following ingredients. First, we have the sum

$$\frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \cdots + \frac{2^n}{3} + \cdots + \frac{2^{50}}{3} = \frac{2^{51}-1}{3}.$$

Second, we have the sum of numbers $-\frac{1}{3}$ repeated as many times as there are even numbers from 0 to 50, i.e. 26 times which adds up to $\frac{-26}{3}$. Third, we have the sum of numbers $-\frac{2}{3}$ repeated as many times as there are odd numbers from 0 to 50, i.e. 25 times which adds up to $\frac{-50}{3}$. Therefore the entire sum in question is

$$\frac{2^{51}-77}{3} = \frac{2^{51}-2}{3} - 25$$

and the computations show that this equals

$$\frac{2,251,799,813,685,248-2}{3} - 25 = 750,599,937,895,057.$$

The answer is $\mathbf{\frac{2^{51}-77}{3}}$, equivalently $\mathbf{\frac{2^{51}-2}{3}} - \mathbf{25}$, and numerically $\mathbf{\frac{2,251,799,813,685,248-2}{3}} - \mathbf{25} = \mathbf{750,599,937,895,057}$. □

PROBLEM 7 (250 pts) Jack and Jill play the following game. Jack writes a sign (either + or -) on a large board. Then Jill writes a number from 1 through 2021 next to it, and so it. This is repeated 2021 times until the collection of numbers from 1 through 2021 is exhausted (Jill can use each number only once). If the resulting sum is x, then Jill gets $\$|x|$ award. What maximal award \$A can Jill **guarantee** to receive? The justification must include Jack's strategy preventing Jill from getting more than \$A and Jill's strategy guaranteeing that at the end she will get at least \$A.

Solution: We will prove that the answer is \$2021. Denote by s_n the sum of the numbers on the board obtained after n steps. In particular $x = s_{2021}$. First we show that Jack can always play so that Jill will not win more than \$2021. Here is his strategy.

1. He begins with any sign.
2. On each step n if $s_n > 0$ he writes -, if $s_n < 0$ he writes +, and if $s_n = 0$ he writes either + or -.
3. Observe that $|s_1| \leq 2021$. Suppose that $|s_n| \leq 2021$ and show that $|s_{n+1}| \leq 2021$. Indeed, this is obvious if $s_n = 0$. Otherwise the choice of the next sign by Jack guarantees that the following inequalities

hold: if $s_n > 0$ then $s_n - 2021 \leq s_{n+1} < s_n$, and if $s_n < 0$ then $s_n < s_{n+1} \leq s_n + 2021$. It follows that $|s_{n+1}| \leq 2021$, and we can continue. In the end this will imply that $|x| = |s_{2021}| \leq 2021$ as desired.

Now let us describe the strategy by Jill that guarantees that she will always be able to win at least \$2021. To this end partition the set of integers $B = \{1, 2, \ldots, 2020\}$ in two subsets:

$$B^+ = \{1, 4, 5, 8, \ldots, 4k-3, 4k, \ldots, 2017, 2020\}$$

and

$$B^- = \{2, 3, 6, 7, \ldots, 4k-2, 4k-1, \ldots, 2018, 2019\}$$

and observe, to begin with, that the sum of elements of B^+ equals the sum of elements of B^-. Now, Jill always writes the next by increasing number from B^+ if the current sign is +, and the next number from B^- if the current sign is -. Doing so, she is waiting for one of the sets B^+ or B^+ to be exhausted. Once one of them is exhausted, she is waiting for the sign from this set to be written for the first time, and once it is written she writes 2021. For example, if the set B^+ was exhausted first, she will wait for + to show for the first time after this, and then she will write 2021. If this sign is never written, then she keeps exhausting the other set and in the end writes 2021. It follows that with this strategy \$A is at least \$2021 as claimed.

The answer is **\$2021**. □