**NAME:**______________________________

**GRADE:**______________________________

**SCHOOL NAME:**____________________

# 2015-2016 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached. Please clearly indicate which problems you are solving. Problems are included on pages 2 and 3.

**PROBLEM 1** (20 pts) Suppose that $a_1, ..., a_{37}$ are integers which are not necessarily distinct. If $\sum_n^{n+5} a_i = 37$ for every $1 \leq n \leq 32$, and $a_1 = 5$, what is $a_{37}$?

*YOUR ANSWER:*

**PROBLEM 2** (30 pts) Find the least positive integer which gives remainder 1 when divided by 2, remainder 2 when divided by 3, remainder 3 when divided by 4, remainder 4 when divided by 5 and remainder 5 when divided by 6.

*YOUR ANSWER:*

**PROBLEM 3** (40 pts) Alex, Bob and Claude major in math, physics and chemistry. Each student majors in exactly one subject, and each subject is a major of exactly one student. We do not know who majors in what, but we know the following. If Alex majors in math, then Claude does not major in physics. If Bob does not major in physics, then Alex majors in math. If Claude does not major in math, then Bob majors in chemistry. You have to figure out, for Alex, Bob, and Claude, in which subject each majors.

*YOUR ANSWER:*

**PROBLEM 4** (50 pts) Humpty and Dumpty have eaten a barrel of strawberry jam and a basket of cookies. First Humpty was eating cookies while Dumpty was eating jam, and then at some point Humpty started eating jam while Dumpty began eating cookies. They started eating at the same time, finished eating at the same time and switched what they were eating at the same time. It is known that Dumpty ate both jam and cookies three times faster than Humpty. Moreover, they ate the same number of cookies. How much more jam has Dumpty eaten compare with Humpty?

*YOUR ANSWER:*

over, please

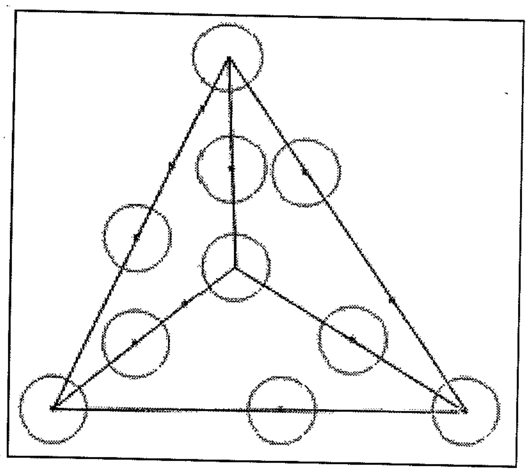
**PROBLEM 5** (70 pts) John has been studying at UAB for 5 years. Overall he has taken 31 exams. In any given year he took more exams than in the previous year. Moreover, in the fifth year he took three times more exams than in the first year. How many exams did John take in the fourth year of his studies?

*YOUR ANSWER:*

**PROBLEM 6** (90 pts) Find all 2-digit integers equal to 1.5 times the product of their digits.

*YOUR ANSWER:*

**PROBLEM 7** (120 pts) Can one put the ten digits from 0 to 9 in the circles below so that the sum along each of the six segments is the same? Completely justify your answer!



*YOUR ANSWER:*

**PROBLEM 8** (150 pts) Consider the set $A$ of all integers from 0 to 399 inclusively; all numbers are written in three-digit format (so instead of 0 we write 000, instead of 17 we write 017 etc). Two distinct numbers from $A$ are said to be *neighbors* if they differ at exactly one place. What is the maximum number of different numbers from $A$ such that no two of them are neighbors?

*YOUR ANSWER:*

# 2015-2016 UAB MTS: SOLUTIONS

**PROBLEM 1** (20 pts) Suppose that $a_1, ..., a_{37}$ are integers which are not necessarily distinct. If $\sum_n^{n+5} a_i = 37$ for every $1 \leq n \leq 32$, and $a_1 = 5$, what is $a_{37}$?

*Solution:* We claim that $a_n = a_{n+6}$ for every $n = 1, \ldots, 31$. Indeed, $\sum_n^{n+5} a_i = 37$ for every $1 \leq n \leq 32$. Hence $a_n = 37 - \sum_{n+1}^{n+5} a_i$ and $a_{n+6} = 37 - \sum_{n+1}^{n+5} a_i$ so that $a_n = a_{n+6}$ as claimed. For instance, $a_1 = 37 - a_2 - a_3 - a_4 - a_5 - a_6 = a_7$ because $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 37$. We conclude that $a_1 = a_7 = a_{13} = a_{19} = a_{25} = a_{31} = a_{37}$.

The answer is $\mathbf{a_{37} = 5}$. □

**PROBLEM 2** (30 pts) Find the least positive integer which gives remainder 1 when divided by 2, remainder 2 when divided by 3, remainder 3 when divided by 4, remainder 4 when divided by 5 and remainder 5 when divided by 6.

*Solution:* Denote the desired number by $x$. Then it follows that $x + 1$ is the least common multiple of $2, 3, 4, 5$ and 6. Therefore $x + 1 = 60$.

The answer is **59**. □

**PROBLEM 3** (40 pts) Alex, Bob and Claude major in math, physics and chemistry. Each student majors in exactly one subject, and each subject is a major of exactly one student. We do not know who majors in what, but we know the following. If Alex majors in math, then Claude does not major in physics. If Bob does not major in physics, then Alex majors in math. If Claude does not major in math, then Bob majors in chemistry. You have to figure out, for Alex, Bob, and Claude, in which subject each majors.

*Solution:* If Bob does not major in physics, then Alex majors in math which implies that Claude does not majors in physics. In other words, in this case nobody majors in physics, a contradiction. Hence Bob majors in physics. Since we know that if Claude does not major in math, then Bob majors in chemistry, we conclude that Claude majors in math. It follows that Alex majors in chemistry.

The answer is: **Alex majors in chemistry, Bob majors in physics, Claude majors in math.** □

**PROBLEM 4** (50 pts) Humpty and Dumpty have eaten a barrel of strawberry jam and a basket of cookies. First Humpty was eating cookies while Dumpty was eating jam, and then at some point Humpty started eating jam while Dumpty began eating cookies. They started eating at the same time, finished eating at the same time and switched what they were eating at the same time. It is known that Dumpty ate both jam and cookies three times faster than Humpty. Moreover, they ate the same number of cookies. How much more jam has Dumpty eaten compare with Humpty?

*Solution:* Since Humpty and Dumpty ate the same number of cookies, it follows that if Dumpty spent, say, $c_D$ minutes eating cookies while Humpty spent $3c_D$ minutes eating cookies. The amount of time Humpty was eating cookies equals the amount of time Dumpty was eating jam. Hence Dumpty was eating jam for $3c_D$ minutes. The amount of time Humpty was eating jam equals the amount of time Dumpty was eating cookies. Hence Humpty was eating jam for $c_D$ minutes. Since Dumpty eats jam three times as fast as Humpty, it follows that the amount of jam eaten by Dumpty is nine times greater than the amount of jam eaten by Humpty.

The answer is **9 times more.** □

**PROBLEM 5** (70 pts) John has been studying at UAB for 5 years. Overall he has taken 31 exams. In any given year he took more exams than in the previous year. Moreover, in the fifth year he took three times more exams than in the first year. How many exams did John take in the fourth year of his studies?

*Solution:* Let the numbers of exams taken by John in the first, second, third, fourth and fifth years at UAB be $x_1, x_2, x_3, x_4$ and $x_5$ respectively. Then it follows that $x_1 < x_2 < x_3 < x_4 < x_5$, $x_1+x_2+x_3+x_4+x_5 = 31$ and $x_5 = 3x_1$. Let us consider possible values of $x_1$. First observe that the inequalities given in the problem can be specified because all numbers $x_1, 1 \leq i \leq 5$, are integers. Thus, we have that

$$x_1 < x_1 + 1 \leq x_2 < x_2 + 1 \leq x_3 < x_3 + 1 \leq x_4 < x_4 + 1 \leq x_5 = 3x_1.$$

It follows that $x_1 + 2 \leq x_3, x_1 + 3 \leq x_4, x_1 + 4 \leq x_5$. Hence $x_1 + (x_1 + 1) + (x_1 + 2) + (x_1 + 3) + (x_1 + 4) = 5x_1 + 10 \leq 31$ and so $x_1 \leq 4$. Suppose that $x_1 = 4$. Then $x_5 = 12$ and so $x_1 + x_2 + x_3 + x_4 = 31 - 12 = 19$. On the other hand by the above this implies that $x_1 + (x_1 + 1) + (x_1 + 2) + (x_1 + 3) = 4x_1 + 6 = 22 \leq x_1 + x_2 + x_3 + x_4 = 19$, a contradiction.

Assume now that $x_1 = 2$. Then $x_5 = 6$, and inequalities above imply that $x_2 = 3, x_3 = 4, x_4 = 5$. Hence $x_1 + x_2 + x_3 + x_4 + x_5 = 2 + 3 + 4 + 5 + 6 = 20$,a contradiction. Moreover, $x_1 = 1$ is impossible because then $x_1 + 4 \leq x_5 = 3x_1 = 3$, a contradiction. Hence $x_1 = 3$ and $x_5 = 9$. It follows that $x_2 + x_3 + x_4 = 31 - 9 - 3 = 19$. Now, if $x_4 \leq 7$ then the greatest possible value of $x_2 + x_3 + x_4$ is $5 + 6 + 7 = 18$, a contradiction. Hence $x_4 = 8$.

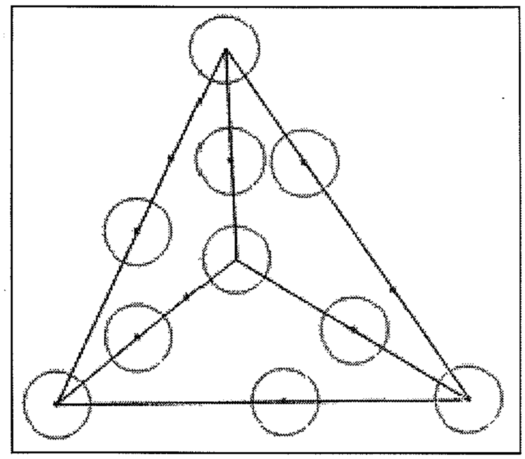
The answer is $\mathbf{x_4 = 8}$. □

**PROBLEM 6** (90 pts) Find all 2-digit integers equal to 1.5 times the product of their digits.

*Solution:* Denote the unknown number by $10x + y$ where $1 \leq x \leq 9, 0 \leq y \leq 9$ are integers. Then it follows that $10x + y = 3xy/2$ and so $20x + 2y = 3xy$. Hence $20x \leq 3xy$ which implies that $20 \leq 3y$ and so $y$ equals $7, 8$ or $9$. If $y = 7$ then $20x + 14 = 21x$ and so $x = 14$, a contradiction. If $y = 8$ then $20x + 16 = 24x$ and hence $x = 4$, so 48 solves the problem. If $y = 9$ then $20x + 18 = 27x$ and $x = 18/7$, a contradiction.

The answer is **48**. □

**PROBLEM 7** (120 pts) Can one put the ten digits from 0 to 9 in the circles below so that the sum along each of the six segments is the same? Completely justify your answer!



*Solution:* Suppose that this is possible. Denote the sum of integers along any segment by $x$. If we add up all numbers along all six segments we will get $6x$. On the other hand, let the sum of the center number and the corner numbers is $y$ while the sum of all other numbers is $z$. Then $y+z=0+1+2+\cdots+9=45$. On the other hand, the center number and the corner numbers are counted three times in the sum of numbers along all segments. It follows that $3y+z=6x=2y+y+z=2y+45$. However, this implies that $45=6x-2y$, a contradiction as an odd number cannot be equal an even number.

The answer is as follows: **it is impossible to put the ten digits from 0 to 9 in the circles so that the sum along each of the six segments is the same.** □

**PROBLEM 8** (150 pts) Consider the set $A$ of all integers from 0 to 399 inclusively; all numbers are written in three-digit format (so instead of 0 we write 000, instead of 17 we write 017 etc). Two distinct numbers from $A$ are said to be *neighbors* if they differ at exactly one place. What is the maximum number of different numbers from $A$ such that no two of them are neighbors?

*Solution:* Divide the set in groups of ten numbers as follows: $0,\ldots,9$, $10,\ldots,19$, $\ldots$, $390,\ldots,399$. Clearly, there are 40 groups of numbers in this partition. Any two number from one group of ten are neighbors. Therefore any collection of numbers no two of whom are neighbors can have at most one number in each group; this implies that the maximal such collection consists of at most 40 numbers.

To show that a desired maximal collection of 40 numbers exists, let us choose from each group a number whose sum of digits is a multiple of 10 (it is easy to see that this is possible). We claim that no two distinct numbers from that collection are neighbors. Indeed, suppose otherwise. Then we may assume that $x$ and $y$ from our collection of numbers are neighbors. It follows that the difference of the sums of digits of $x$ and $y$ is positive but at most 9. Clearly it is impossible with our choice of numbers as each number has the sum of digits which is a multiple of 10.

The answer is **40**. □