UNIVERSITY OF ALABAMA SYSTEM
JOINT DOCTORAL PROGRAM IN APPLIED
MATHEMATICS
JOINT PROGRAM EXAMINATION
Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May, 2000

Instructions: Do 7 of the 8 problems for full credit. Be sure to indicate which 7 are to be graded. Include all work. Write your student ID number on every page of your exam.
1. Let $V$ be a vector space of finite dimension $n$, let $T$ be a linear operator on $V$ with $k + 1$ distinct eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k$, and let the eigenspace corresponding to $\lambda_0$ have dimension $n - k$. Prove that the operator $T^m$ is diagonalizable for each positive integer $m$.

2. Let $V$ and $W$ be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation of rank $r$ where $1 \leq r < \min\{\dim(V), \dim(W)\}$. Prove that there exist bases $\alpha$ for $V$ and $\beta$ for $W$ such that the matrix representation for $T$ with respect to $\alpha$ and $\beta$ has the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

3. Let $V$ be a finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle$, and let $T$ be a self-adjoint operator on $V$. Prove that there exists a self-adjoint operator $S$ on $V$ such that $T = S^2$ if and only if $\langle Tx, x \rangle \geq 0$ for all $x \in V$.

4. (a) Let $A$ be a $10 \times 10$ complex matrix with characteristic polynomial $C_A(x) = (x - 1)^6(x + 2)^4$, minimal polynomial $M_A(x) = (x - 1)^3(x + 2)^2$, and $\dim E_1 = 3$, $\dim E_{-2} = 2$, where $E_1$ and $E_{-2}$ are the eigenspaces corresponding to the eigenvalues $1$ and $-2$ respectively. Find a Jordan canonical form of $A$.

(b) Let $A$ be an $8 \times 8$ complex matrix with characteristic polynomial $C_A(x) = (x + i)^3(x - i)^3(x - 1)^2$, and $\dim E_{-i} = \dim E_i = \dim E_1 = 2$. Find the minimal polynomial of $A$.

5. (a) Calculate $A^{-1}$ and $\kappa_\infty(A)$ for the matrix

\[
A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}.
\]

(b) For the above $A$, find $b$, $\delta b$ and $x$, $\delta x$ such that

$A x = b$, \quad $A(x + \delta x) = b + \delta b$

with $\| \delta b \|_\infty / \| b \|_\infty$ small and $\| \delta x \|_\infty / \| x \|_\infty$ large.

(c) Let $A \in \mathbb{R}^{n \times n}$ be given, nonsingular, and consider the linear system problem

$A x = b,$

where $b \in \mathbb{R}^n$ is given. Let $x + \delta x \in \mathbb{R}^n$ be an approximate solution to this problem, satisfying

$A(x + \delta x) = b + \delta b.$

Prove that

$\frac{\| \delta x \|}{\| x \|} \leq \kappa(A) \frac{\| \delta b \|}{\| b \|}$

and comment on the significance of this result.
6. Use a $QR$ decomposition, with exact arithmetic, to solve the least squares problem for the overdetermined system

\[
\begin{bmatrix}
1 & -3 \\
2 & 4 \\
2 & 5
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
4 \\
3 \\
-5
\end{bmatrix}.
\]

State the magnitude of the minimum residual.

7. Let $A \in \mathbb{R}^{n \times n}$ be given, and let $Q_0$ be an arbitrary $n \times n$ orthogonal matrix. Consider the sequence of matrices $R_k$ and $Q_k$ computed as follows:

\[
Z_{k+1} = AQ_k,
Q_{k+1}R_{k+1} = Z_{k+1}.
\]

In the last step, we compute the $QR$ decomposition of $Z_{k+1}$ to get $Q_{k+1}$ and $R_{k+1}$. Assume that

\[
\lim_{k \to \infty} Q_k = Q_\infty
\]

and

\[
\lim_{k \to \infty} R_k = R_\infty
\]

exist. Prove that the eigenvalues of $A$ are given by the diagonal elements of $R_\infty$.

8. Let $A \in \mathbb{C}^{n \times n}$ be given, Hermitian, and let $(\lambda, u)$ be an arbitrary eigenpair of $A$, with $u$ real and $\|u\|_2 = 1$. Let $x \approx u$ be given, with $\|x\|_2 = 1$ and define $\sigma$ by

\[
\sigma = \frac{(Ax, x)}{(x, x)},
\]

the Rayleigh quotient of $x$. Prove that

\[
|\lambda - \sigma| \leq C\|u - x\|_2^2.
\]