Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number on every page of your exam.
1. Let $A = I + x \cdot y^*$, where $x, y \in \mathbb{C}^m (\neq 0)$ and $I$ is the $m \times m$ identity matrix.

(a) Determine a necessary and sufficient condition on $x, y$ so that $A$ admits an eigenvalue decomposition. Then find such a decomposition.

(b) Determine a necessary and sufficient condition on $x, y$ so that $A$ admits an unitary diagonalization. Then find such a diagonalization.

2. Let $A, E \in \mathbb{R}^{m \times m}$ with $E \neq 0$ and $(A + E)$ being singular.

(a) Prove $\text{cond}(A) \geq \|A\|/\|E\|$ for any matrix norm consistent with some vector norm.

(b) Suppose $A$ is non-singular and $y \in \mathbb{R}^m$ is non-trivial satisfying $\|A^{-1}\|_2\|y\|_2 = \|A^{-1}y\|_2$.

Show that equality holds in the relation (a) for the 2-norm for $E = -yx^T/\|x\|_2^2$, $x = A^{-1}y$.

(c) Use the inequality in (a) to get a lower bound for $\text{cond}_\infty(A) = \|A\|_\infty\|A^{-1}\|_\infty$ for the matrix $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & \epsilon & \epsilon \\ 1 & \epsilon & \epsilon \end{pmatrix}$ where $0 < \epsilon < 1$.

3. Let $A \in \mathbb{R}^{n \times m}$ with $\text{rank}(A) = r \geq 0$.

(a) Show that for every $\epsilon > 0$, there exists a full rank matrix $A_\epsilon \in \mathbb{R}^{n \times m}$ such that $\|A - A_\epsilon\| < \epsilon$.

(b) Assume $r > 0$ and let $A = U\Sigma V^T$ be a SVD of $A$, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. For each value $k = 0, 1, 2, \cdots, r - 1$, define $A_k = U\Sigma_k V^T$ where $\Sigma_k$ is the upper-left $k \times k$ sub-matrix of $\Sigma$. Show that

(i) $\sigma_{k+1} = \|A - A_k\|_2$.

(ii) $\sigma_{k+1}$ = \min\{\|A - B\|_2 : B \in \mathbb{R}^{n \times m}$ and $\text{rank}(B) \leq k\}.$

4. Let $A_1, A_2, \ldots, A_k \in F^{n \times n}$ such that $A_i$ has $n$ distinct eigenvalues. Prove that there exists an invertible $P \in F^{n \times n}$ such that $P^{-1}A_jP$ is a diagonal matrix for each $1 \leq j \leq k$ if and only if $A_iA_j = A_jA_i$ for all $1 \leq i, j \leq k$. 

5. (a) Let \( x, y \in \mathbb{R}^n \) such that \( x \neq y \) but \( \|x\|_2 = \|y\|_2 \). Show that there exists a reflector \( Q \) of the form \( Q = I - 2uu^T \), where \( u \in \mathbb{R}^n \) and \( \|u\|_2 = 1 \) such that \( Qx = y \).

(b) Let \( A = \begin{bmatrix} 4 & 4 & 1 \\ 3 & -2 & 7 \\ 0 & 3 & 1 \end{bmatrix} \). Use the Householder reflector to find an QR factorization for the matrix \( A \), i.e., \( A = QR \) where \( Q \) is an orthogonal matrix and \( R \) is an upper triangular matrix.

6. Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 3 \\ 1 & 0 \end{pmatrix} \).

(a) Find an QR factorization of \( A \) by the Gram-Schmidt process.

(b) Use the QR factorization from (a) to find the best least square fit by a linear function for \((1, -2), (-2, 0), (3, 2)\) and \((0, 3)\).

7. For which positive integers \( n \) does there exist \( A \in \mathbb{R}^{n \times n} \) such that \( A^2 + A + I = O \). Justify your claim.

8. (In this problem, you may use Schur’s factorization without proof).

(a) Let \( A \in \mathbb{C}^{m \times m} \). Show that \( A \) is normal (i.e., \( AA^* = A^*A \)) if and only if there is an unitary matrix \( V \) such that \( A = A^*V \).

(b) Assume that \( A \) is normal. Show that all eigenvalues of \( A \) are real if and only if \( A \) is hermitian.