UNIVERSITY OF ALABAMA SYSTEM
Joint Doctoral Program in Applied Mathematics
Joint Program Exam: Linear Algebra and Numerical Linear Algebra

TIME: THREE AND ONE HALF HOURS

May 2006

Instructions: Do 7 of the 8 problems for full credit. Include all work. Write your student ID number, and problem number, on every page.

1. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Suppose x and x_c satisfy, respectively,

$$Ax = \mathbf{b}$$

and

$$(A + \Delta A)x_c = \mathbf{b}$$

where ΔA is the change in A and $\mathbf{b}$ is a nonzero vector.

(a) Show that if $\|A^{-1}\Delta A\| < 1$, then

$$\frac{\|x - x_c\|}{\|x\|} \leq \frac{\|A^{-1}\Delta A\|}{1 - \|A^{-1}\Delta A\|}.$$

(b) Show that if $\|A^{-1}\| \cdot \|\Delta A\| < 1$, then

$$\frac{\|x - x_c\|}{\|x\|} \leq \frac{\kappa(A)\frac{\|\Delta A\|}{\|A\|}}{1 - \kappa(A)\frac{\|\Delta A\|}{\|A\|}}$$

2. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues such that $|\lambda_1| > |\lambda_2| \geq \cdots |\lambda_{n-1}| > |\lambda_n| > 0$. Suppose $\mathbf{z} \in \mathbb{R}^n$ with $\mathbf{z}^T\mathbf{x}_1 \neq 0$, where $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$. Prove that, for some constant C,

$$\lim_{k\to\infty} \frac{A^k\mathbf{z}}{\lambda_1^k} = C\mathbf{x}_1$$

and use this result to devise an algorithm for computing λ_1 and $\mathbf{x}_1$. Explain how the calculation should be modified to obtain (a) λ_n and (b) the simple eigenvalue closest to 5.

3. Let $A \in \mathbb{C}^{n\times n}$ be normal. Show that:

(a) A is Hermitian if and only if its eigenvalues lie on the real axis.

(b) A is skew Hermitian (i.e., $A^* = -A$ where A^* denotes the conjugate transpose of the matrix A) if and only if its eigenvalues lie on the imaginary axis.

(c) A is unitary if and only if its eigenvalues lie on the unit circle.

4. Let $A \in \mathbb{C}^{n\times n}$ be tridiagonal and Hermitian, with all its super-diagonal entries nonzero. Prove that the eigenvalues of A are distinct.
(Hint: Show that for any scalar λ, the matrix $A - \lambda I$ has rank at least $n - 1$.)

5. Assume that $A \in \mathbb{C}^{m\times n}$.

(a) Prove that A and A^*A have the same null space.

(b) Use (a) to show that A^*A is nonsingular if and only if the rank of A is n.

6. (a) Let

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Find a matrix T such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.

(b) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is 4.

7. Given a matrix $A \in \mathbb{C}^{n\times n}$, a number $z \in \mathbb{C}$ that is *not* an eigenvalue of A, a positive number ε, and denoting σ_n the smallest singular value of $zI - A$, prove that the following conditions are equivalent:

(a) z is an eigenvalue of $A + B$ for some B with $\|B\|_2 \leq \epsilon$;

(b) there exists $u \in \mathbb{C}^n$ with $||(A - zI)u||_2 \leq \varepsilon$ and $||u||_2 = 1$;

(c) $\sigma_n \leq \varepsilon$;

(d) $||(zI - A)^{-1}||_2 \geq 1/\varepsilon$.

8. Suppose $A \in \mathbb{C}^{n\times n}$ is diagonalizable with $A = V\Lambda V^{-1}$, where Λ is a diagonal matrix, and let $B \in \mathbb{C}^{n\times n}$ be arbitrary. Then prove that every eigenvalue of $A+B$ lies in at least one of n disks in the complex plane of radius $\kappa_2(V)\ \|B\|_2$ centered at the eigenvalues of A, where κ_2 is the 2-norm condition number.
(Hint: You may use the known fact (see the previous problem) that if z is an eigenvalue of $A+B$ but not an eigenvalue of A, then $\|(zI - A)^{-1}\|_2 \geq 1/\|B\|_2$)