University of Alabama System

Joint Ph.D Program in Applied Mathematics

Joint Program Exam: Linear Algebra and Numerical Linear Algebra

September 2013

Instructions:

• This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.

• Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.

• Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.

• Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.

• The use of calculators or other electronic gadgets is not permitted during the exam.

• Write legibly using dark pencil or pen.
1. Let $T$ be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ defined by

$$T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e).$$

(a) Find the characteristic and minimal polynomial of $T$.
(b) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of $T$.
(c) Find the Jordan form of $T$ with respect to your basis.

2. Let $U$ and $V$ be subspaces of the finite dimensional inner product space $V$.

(a) Prove that $U^\perp \cap W^\perp = (U + W)^\perp$.
(b) Prove that $\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp)$.

3. Let $B \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Let $\lambda_1$ be the maximum of the eigenvalues of $B$. For $0 \neq x \in \mathbb{R}^n$, using the usual 2-norm $\|x\|_2$, define the Raleigh quotient $\rho_B(x)$ for $B$ by

$$\rho_B(x) = \frac{(Bx, x)}{(x, x)} = \frac{x^t B x}{\|x\|_2^2}.$$

Prove the following:

(a) If $B$ and $\lambda_1$ are defined as above, prove that $\lambda_1 = \max\{\rho_B(x) : x \in \mathbb{R}^n \text{ and } \|x\|_2 = 1\}$
(b) Let $A \in \mathbb{R}^{n \times n}$ be a matrix with largest singular value $\sigma_1$. If

$$\|A\|_2 = \max\{\|Ax\|_2 : x \in \mathbb{R}^n \text{ and } \|x\|_2 = 1\}$$

show that $\|A\|_2 = \sigma_1$.

4. Let $A = A^{(1)}$ be strictly column diagonally dominant. After one step of Gauss elimination $A^{(1)}$ is reduced to

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & A_{s}^{(2)} \end{pmatrix}.$$

Show that $A_{s}^{(2)} \in \mathbb{R}^{(n-1) \times (n-1)}$ is also strictly column diagonally dominant. (Therefore, if LU decomposition with partial pivoting is applied to $A$, no row interchanges take place.)
5. Let $A_1, A_2, \ldots, A_k \in \mathbb{R}^{n \times n}$ such that $A_1$ has $n$ distinct eigenvalues. Prove that there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that $P^{-1} A_j P$ is a diagonal matrix for each $1 \leq j \leq k$ if and only if $A_i A_j = A_j A_i$ for all $1 \leq i, j \leq k$.

6. Let $V$ be a finite dimensional inner product space and $W \subset V$ a subspace. For every $v \in V$ there is a unique decomposition $v = w + w'$ with $w \in W$ and $w' \in W^\perp$. Define a map $T : V \to V$ by $Tv = w - w'$. Prove that $T$ is a unitary and self-adjoint operator.

7. Let $A$ be an $n \times n$ real matrix of full rank with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and let $X$ be a matrix that diagonalizes $A$, i.e. $X^{-1} A X = D$ where $D$ is a diagonal matrix. If $A' = A + E$ and $\lambda'$ is an eigenvalue of $A'$, prove that

$$\min_{1 \leq i \leq n} |\lambda' - \lambda_i| \leq \kappa_2(X) \|E\|_2$$

where $\kappa_2(X)$ is the 2-norm condition number of $X$.

8. Given the data $(0, 1), (3, 4), (6, 5)$. Use a QR factorization technique to find the best least squares fit by a linear function.