The University of Alabama System
Joint Ph.D Program in Applied Mathematics
Linear Algebra and Numerical Linear Algebra JP Exam
September 2016

Instructions:

• This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do seven of the eight problems for full credit.

• Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.

• Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.

• Begin each solution on a new page and write the last four digits of your university student ID number, and problem number, on every page. Please write only on one side of each sheet of paper.

• The use of calculators or other electronic gadgets is not permitted during the exam.

• Write legibly using dark pencil or pen.
1. Let \( a, b \in \mathbb{R} \) such that \( a \neq b \). Let \( A \in \mathbb{R}^{6\times6} \) such that the characteristic polynomial of \( A \) is \( C(x) := (x - a)^4(x - b)^2 \) and the minimal polynomial of \( A \) is \( m(x) := (x - a)^2(x - b) \). Describe all possible Jordan forms for \( A \).

2. Let \( V \) be a finite dimensional real vector space. Let \( W_1 \) and \( W_2 \) be subspaces of \( V \). We define the following operations:

\[
(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2')
\]
and

\[
\alpha \ast (w_1, w_2) := (\alpha w_1, \alpha w_2)
\]

for all \((w_1, w_2) \in W_1 \times W_2\) and \((w_1', w_2') \in W_1 \times W_2\) and all \(\alpha \in \mathbb{R}\). The set \( W_1 \times W_2 \) is a vector space with respect to these operations.

(a) Let \( U := \{(u, -u) : u \in W_1 \cap W_2\} \). Prove that \( U \) is a subspace of \( W_1 \times W_2 \). Also prove that \( U \) is isomorphic to \( W_1 \cap W_2 \).

(b) Define the map \( T : W_1 \times W_2 \rightarrow W_1 + W_2 \) by \( T(w_1, w_2) := w_1 + w_2 \). Prove that \( T \) is a linear transformation.

(c) Use the above to prove that \( \dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) \).

3. Let \( P_2[0,2] \) represent the set of all polynomials with real coefficients and of degree less than or equal to 2, defined on \([0,2]\). For \( p := (p(t)) \in P_2 \) and \( q := (q(t)) \in P_2 \), define

\[
\langle p, q \rangle := p(0)q(0) + p(1)q(1) + p(2)q(2)
\]

(a) Verify that \( \langle p, q \rangle \) is an inner product.

(b) Let \( T \) represent the linear transformation that maps an element \( p \in P_2 \) to the closest element of the span of the polynomials 1 and \( t \) in the sense of the norm associated with the inner product. Find the matrix \( A \) of \( T \) in the standard basis \( \{1, t, t^2\} \) of \( P_2 \).

(c) Is \( A \) symmetric? Is \( T \) self-adjoint? Do these contradict each other?

(d) Find the minimal polynomial of \( T \).

4. Suppose that \( T \) is a linear map from a vector space \( V \) to \( \mathbb{F} \) where \( \mathbb{F} \) can be either \( \mathbb{R} \) or \( \mathbb{C} \). Prove that if a vector \( u \) in \( V \) is not in \( \text{null}(T) \), then

\[
V = \text{null}(T) \oplus \{\alpha u : \alpha \in \mathbb{F}\}
\]

where \( \text{null}(T) \) is the null space of \( T \).
5. (a) Suppose \( p, q \in \mathbb{R} \) with \( p \) and \( q \) positive and \( 1/p + 1/q = 1 \). Show that for any matrix \( A \in \mathbb{C}^{n \times n} \), we have \( \|A\|_p = \|A^*\|_q \), where \( A^* \) is the conjugate transpose of \( A \). Here \( \|A\|_p \) denotes the matrix \( p \)-norm induced by the vector \( p \)-norm defined by \( \|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p} \).

(b) Prove that
\[
\|A\|_2^2 \leq \|A\|_p \|A\|_q
\]
for any \( A \in \mathbb{C}^{n \times n} \) and any positive \( p \) and \( q \in \mathbb{R} \) with \( 1/p + 1/q = 1 \).

(c) Prove that for any \( p \geq 1 \) and any diagonal matrix \( D \in \mathbb{C}^{n \times n} \), we have
\[
\|D\|_p = \max\{|d_{ii}| : 1 \leq i \leq n\}.
\]

(d) Show that \( \|A\|_2 \) is the largest singular value of \( A \).

6. (a) Let
\[
x := \begin{bmatrix} 1 \\ 7 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \quad y := \begin{bmatrix} -4 \\ 4 \\ 0 \\ -4 \end{bmatrix}.
\]
Is there an orthogonal matrix \( Q \) so that \( Qx = y \)? If so, use EXACT arithmetic to find it. If not, explain why.

(b) Let
\[
A := \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad b := \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}.
\]
Compute a QR decomposition of \( A \) using Householder reflections and then solve the least square problem \( \min_x \|b - Ax\|_2 \) using the QR decomposition.

7. For the matrix \( A := \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & \sqrt{3}/2 \\ -1/2 & \sqrt{3}/2 & 1 \end{pmatrix} \), obtain the singular value decomposition of \( A \) (in the form \( A = U \Sigma V^T \) where \( U \) and \( V \) are orthogonal and \( \Sigma \) is diagonal). Use this to find the Frobenius norm \( \|A\|_F \) and the 2-norm \( \|A\|_2 \).

8. Suppose that \( A \in \mathbb{C}^{n \times n} \) is normal, i.e., \( AA^* = A^*A \). Show that if \( A \) is also upper triangular, it must be diagonal. Use this to show that \( A \) is normal if and only if it has \( n \) orthonormal eigenvectors.