The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

May 2024

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let T be a linear operator on a vector space V, $dim(V) = n$.

   (a) If for some vector $\mathbf{v}$, the vectors $\mathbf{v}$, $T(\mathbf{v})$, $T^2(\mathbf{v})$, ..., $T^{n-1}(\mathbf{v})$ are linearly independent, show that every eigenvalue of T has only one corresponding eigenvector upto a scalar multiplication.

   (b) If T has n distinct eigenvalues, and vector $\mathbf{u}$ is the sum of n eigenvectors corresponding to the distinct eigenvalues, show that the vectors $\mathbf{u}$, $T(\mathbf{u})$, $T^2(\mathbf{u})$, ..., $T^{n-1}(\mathbf{u})$ are linearly independent (and thus form a basis of V).

2. A is a real 3×3 matrix, and we know that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}, \quad A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}$$

   (a) What are the eigenvalues and associated eigenvectors of A? Can we use the set of eigenvectors as the basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

   (b) Calculate

$$A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

   (c) Does the linear system $A\mathbf{x} = \mathbf{b}$ have a solution for any $\mathbf{b} \in \mathbb{R}^3$? If so, why? If not, for what kind of $\mathbf{b} \in \mathbb{R}^3$ is $A\mathbf{x} = \mathbf{b}$ solvable?

   (d) Determine whether matrix A has the following properties. Explain your reasoning.
   (i) diagonalizable
   (ii) invertible
   (iii) orthogonal
   (iv) symmetric

3. Let A and B in $\mathbb{R}^{n \times n}$ such that $AB - BA = A$.

   (a) Prove that $A^k B - BA^k = kA^k$

   (b) Prove that A is nilpotent

4. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be positive definite, and suppose A_{11} is $j \times j$ and A_{22} is $k \times k$. Show that A_{22} and $\widehat{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ are both positive definite.

5. Let $H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$, where A is square and $A = U\Sigma V^T$ is its SVD. Let $\Sigma = diag(\sigma_1, ..., \sigma_n), U = [u_1, ..., u_n]$, and $V = [v_1, ..., v_n]$. Find the eigenvalues and corresponding eigenvectors of H in terms of σ_i, u_i, and $v_i, i = 1, ..., n$.

6. Prove the following statements.

   (a) Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Then for every vector $x \in \mathbb{C}^n$, we have $\langle Ax, x\rangle \in \mathbb{R}$.

   (b) Let $r(x) = \frac{\langle Ax,x\rangle}{\langle x,x\rangle} = \frac{x^*Ax}{x^*x}$ with $x \neq 0$, i.e. $r(x)$ is the Rayleigh quotient of x. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Then $r(x) \in \mathbb{R}$ for any non-zero vector $x \in \mathbb{C}^n$.

   (c) If $x \in \mathbb{C}^n$ is an arbitrary unit vector, then $r(x)x = (x^*Ax)x$ is the orthogonal projection of the vector Ax onto the line spanned by x, i.e.
   $$\|Ax - r(x)x\|_2 = \min_{\mu\in\mathbb{C}} \|Ax - \mu x\|_2$$
   where $r(x)$ is the Rayleigh quotient of x defined in part (b).

7. Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix and
   $$\begin{aligned} Ax &= b \\ (A + \Delta A)(x + \Delta x) &= b + \Delta b. \end{aligned}$$
   Assume that $\|\Delta A\|$ is small so that $\|\Delta A\|\|A^{-1}\| < 1$. Show that
   $$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\Delta A\|}{\|A\|}} \left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|b\|} \right)$$
   where $\kappa(A) = \|A\|\|A^{-1}\|$, the condition number of A.

8. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ where A is to be considered as a matrix over $\mathbb{C}$.

(a) Determine the minimal and characteristic polynomials of A and the Jordan form for A.

(b) Determine all generalized eigenvectors of A and a basis $\mathcal{B}$ of $\mathbb{C}^4$ with respect to which the operator $T_A : x \to Ax$ has Jordan form. Use this to write down a matrix P such that $P^{-1}AP$ is in the Jordan form.