The University of Alabama at Birmingham
Department of Mathematics
Qualifying Exam in Linear Algebra and Numerical Linear Algebra

August 11, 2026

Duration: 8:30 a.m.–12:10 p.m. (220 minutes)

Instructions:

1. Print the last 4 digits of your supplier ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. For each problem that you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

3. This is a closed-book examination. Once the exam begins, you have three and one-half hours to do your best. You are required to do **all seven problems for full credit**. In Part I, there are four (4) problems, which must be completed within the first 120 minutes. In Part II, there are three (3) problems, which must be completed within the rest of the time. There is a 10-minute break between Part I and Part II.

4. Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.

5. Justify your solutions: cite theorems that you use, provide counterexamples for disproof, give explanations, and show calculations for numerical problems.

6. The use of calculators or other electronic gadgets is not permitted during the exam.

Part I

1. (a) Let V and W be finite-dimensional vector spaces with $\dim V = \dim W < \infty$. Let $T : V \to W$ and $S : W \to V$ be linear transformations such that $(ST)(x) = x$ for all $x \in V$. Show that T is one-to-one, onto, and hence invertible with $T^{-1} = S$.

   (b) Let $A \in \mathbb{R}^{n\times n}$ satisfy $A^2 = I$, where I is the identity matrix. Show that
$$\operatorname{rank}(A + I) + \operatorname{rank}(A - I) = n.$$

2. (a) Let A and B be normal matrices such that $\operatorname{Im} A \perp \operatorname{Im} B$. Prove that $A + B$ is a normal matrix.

   (b) Let P_1 and P_2 be projections. Prove that $P_1 + P_2$ is a projection if and only if
$$P_1P_2 = P_2P_1 = 0.$$

3. Let V be an n-dimensional complex vector space, S and T be two linear operators on V, and I be the identity map. Prove that there exist ordered bases α and β for V such that the matrix representation of S with respect to α is equal to the matrix representation of T with respect to β if and only if
$$\operatorname{rank}(S - cI)^k = \operatorname{rank}(T - cI)^k$$
for all $c \in \mathbb{C}$ and $k = 1, 2, \ldots, n$.

4. Let $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r$, $0 < r \leq \min\{m, n\}$. Let $A = U\Sigma V^T$ be a singular value decomposition of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. For each value $k = 0, 1, 2, \ldots, r - 1$, show that
$$\sigma_{k+1} = \min\left\{\|A - B\|_2 : B \in \mathbb{R}^{m\times n} \text{ and } \operatorname{rank}(B) \leq k\right\}.$$

Part II

5. Let $A \in \mathbb{C}^{n\times n}$ with $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A.

   (a) Show that $I - A$ is nonsingular.

   (b) Show that the matrix series
   $$\sum_{k=0}^{\infty} A^k$$
   converges and that
   $$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

6. Consider the least squares problem
   $$\min_{\boldsymbol{x}} \|\boldsymbol{b} - A\boldsymbol{x}\|_2,$$
   where A is an ill-conditioned matrix that has full column rank. Let $\boldsymbol{x}$ denote the unique least-squares solution. Consider the regularization approach that replaces the normal equations by the modified, better-conditioned system
   $$(A^T A + \gamma I)\boldsymbol{x}_\gamma = A^T\boldsymbol{b},$$
   where $\gamma > 0$ is a regularization parameter.

   (a) Show that
   $$\kappa_2^2(A) \geq \kappa_2(A^T A + \gamma I).$$

   (b) Show that
   $$\|\boldsymbol{x}_\gamma\|_2 \leq \|\boldsymbol{x}\|_2.$$

7. Let $H \in \mathbb{C}^{n\times n}$ satisfy
   $$H^* = H, \qquad H^* H = I, \qquad H \neq I,$$
   where H^* denotes the conjugate transpose of H. A *Householder reflector* is a matrix of the form
   $$I - 2\boldsymbol{u}\boldsymbol{u}^*,$$
   where $\boldsymbol{u} \in \mathbb{C}^n$ is a unit vector. Prove that the following statements are equivalent:

   (a) H is a Householder reflector.

   (b) $\operatorname{rank}(I - H) = 1$.