The University of Alabama at Birmingham
Department of Mathematics
Qualifying Exam in Linear Algebra and Numerical Linear Algebra

May 13, 2026

Duration: 8:30 a.m.–12:10 p.m. (220 minutes)

Instructions:

1. Print the last 4 digits of your supplier ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. For each problem that you attempt, try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.

3. This is a closed-book examination. Once the exam begins, you have three and one-half hours to do your best. You are required to do **all seven problems for full credit**. In Part I, there are four (4) problems, which must be completed within the first 120 minutes. In Part II, there are three (3) problems, which must be completed within the rest of the time. There is a 10-minute break between Part I and Part II.

4. Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.

5. Justify your solutions: cite theorems that you use, provide counterexamples for disproof, give explanations, and show calculations for numerical problems.

6. The use of calculators or other electronic gadgets is not permitted during the exam.

Part I

1. (a) Let $T : V \to V$ be a linear operator. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are non-zero vectors in V such that $T(\boldsymbol{v}_1) = \boldsymbol{0}$ and $T(\boldsymbol{v}_i) = \boldsymbol{v}_{i-1}$ for $2 \leq i \leq n$. Prove that $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ is a linearly independent set.

   (b) Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis of a vector space V. Let $C = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$ be a linearly independent set in V. Prove that there is an integer k, $1 \leq k \leq n$, such that the vectors $\boldsymbol{u}_k, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$ are linearly independent.

2. Let $\|\cdot\|$ denote any norm of $\mathbb{C}^m$. The corresponding dual norm $\|\cdot\|'$ is denoted by the formula

$$\|\boldsymbol{x}\|' = \sup_{\|\boldsymbol{y}\|=1} |\boldsymbol{y}^* \boldsymbol{x}|,$$

   where $\boldsymbol{y}^*$ is the conjugate transpose of $\boldsymbol{y}$.

   (a) Prove that $\|\cdot\|'$ is a norm.

   (b) Show that for a given $\boldsymbol{x} \in \mathbb{C}^m$, there exists a nonzero $\boldsymbol{z} \in \mathbb{C}^m$ such that

$$|\boldsymbol{z}^* \boldsymbol{x}| = \|\boldsymbol{z}\|' \|\boldsymbol{x}\|.$$

   (c) Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^m$ with $\|\boldsymbol{x}\| = \|\boldsymbol{y}\| = 1$ be given. Show that there exists a rank-one matrix $B = \boldsymbol{y}\boldsymbol{z}^*$ such that $B\boldsymbol{x} = \boldsymbol{y}$ and $\|B\| = 1$, where $\|B\|$ is the matrix norm of B induced by the vector norm $\|\cdot\|$.

3. Suppose that A is a real, $n \times n$ symmetric matrix with $A^3 = A^2 + A - I$. Show that A is invertible and in fact A is its own inverse.

4. Let A be an $n \times n$ real matrix of rank one.

   (a) Show that $A = \boldsymbol{u}\boldsymbol{v}^T$ for some non-zero real vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

   (b) Show that $det(I + \boldsymbol{u}\boldsymbol{v}^T) = 1 + \boldsymbol{u}^T \boldsymbol{v}$, where I is the $n \times n$ identity matrix

Part II

5. Let $A \in \mathbb{C}^{m\times n}$ with $rank(A) = r$.

   (a) Show that A is the sum of r rank-one matrices

   $$A = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^*,$$

   where singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_r\}, \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r\}$ are orthonormal sets.

   (b) For any ν with $0 \leq \nu \leq r$, show that the matrix $A_\nu = \sum\limits_{j=1}^{\nu} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^*$ satisfies

   $$\|A - A_\nu\|_F = \inf_{\substack{B\in\mathbb{C}^{m\times n} \\ rank(B)\leq\nu}} \|A - B\|_F = \sqrt{\sigma_{\nu+1}^2 + \cdots + \sigma_r^2},$$

   where $\|\cdot\|_F$ represents Frobenius norm.

6. Let X be n-dimensional vector space, Y be m-dimensional vector space, and $T \in L(X, Y)$ be a linear transformation of rank r where $1 \leq r < \min\{n, m\}$. Prove that there exist bases α for X and β for Y such that the matrix representation for T with respect to α and β has the form

   $$\begin{bmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

   where $I_{r\times r}$ is the $r \times r$ identity matrix, etc.

7. (a) Let L and G he subspaces of $\mathbb{C}^n$ and $dim(G) > dim(L)$. Then prove that there is a nonzero vector in G orthogonal to L.

   (b) Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Then for every $i = 1, 2, \cdots, n$, show that

   $$\lambda_i = \min_{dim(L)=i} \left(\max_{\boldsymbol{x}\in L\setminus\{\boldsymbol{0}\}} \frac{\boldsymbol{x}^H A \boldsymbol{x}}{\boldsymbol{x}^H \boldsymbol{x}} \right),$$

   where L is a subspace with $dim(L) = i$.