TOPOLOGY NOTES, SPRING, 2020
SYLLABUS AND NOTES, SECTION 1

JOHN C. MAYER

Syllabus

Course:	MA 670-1C	Topology I
Meetings:	MWF 10:10 - 11:00 AM	UH 4002
Instructor:	John Mayer	UH 4022
Email:	jcmayer@uab.edu	
Office Hours:	by appointment	or just drop by and take your chances

Format of Course. This is a "do-it-yourself" course. You are *strongly discouraged* to consult any textbook, internet site, or "expert." You may talk to each other informally and to me, but work presented at the board must be your own or properly credited. It is OK to say "The idea for this proof was suggested to me by ..." and incurs no penalty.

Material. The only material for the course will be this set of **Topology Notes** which will be updated, distributed, and posted on Canvas, periodically. Occasionally, there may be inadvertent errors in the notes. Please point them out and they will be corrected in the next update.

Most items appearing in italics (theorems, lemmas, propositions, corollaries, problems, and examples) are for students to work and present proofs and explanations at the board. Set out items not in italics (definitions, axioms, remarks, conjectures, and questions) are for your use and reference.

Exercises are also in italics, but are not usually to be presented at the board. They may count toward your grade (see "Rules" below). You are responsible for knowing how to solve them. You may use true statements from the exercises in your boardwork without proof.

Rules.

(1) Mere presence counts 1 point per class meeting.
(2) Each item presented at the board and defended correctly counts 2–4 points, depending upon quality of explanation/proof/defense. You may receive an additional point for a comment on another's proof that moves discussion forward.
(3) Priority order for presenting is determined (in order) by:
  (a) Persons with lowest board point total.
  (b) Persons who have not yet presented that day.
  (c) Random experiments to break ties.
(4) Exercises are not presented in class, but may be turned in for a written homework grade (see "Grading" below). (Deadline is first class meeting after we move beyond the subsection in which the exercise appears.) Proof and/or explanation is always required — a bare answer never suffices.

Date: December 5, 2019.
1991 *Mathematics Subject Classification.* Primary: 54F20.
Key words and phrases. topology.

(5) There are two tests: midterm and final (see “Grading” below).

Grading. Items will be weighted the flexible amounts indicated below, by your choice, presumably so as to produce the best individual grade. You can (and may) rely almost entirely on boardwork for your grade.

Boardwork	30–100%	rank-ordered subject to a minimum
Homework	0–20%	rank-ordered subject to a minimum
Tests and Quizzes	0–50%	mid-term and final count equally

You *are not required* to do any written homework unless you want to. It can only help your grade. You *are required* to take the tests and quizzes, though they cannot hurt you much if your boardwork total is excellent. Any day that there is not enough board work to keep the class engaged, there will be a quiz!

Attendance. Attendance in class is required, as you are there not only to present your own results, but to critique the work of others. Participation is expected. Unexcused absence is a 2 point penalty per day on your participation total. Lateness of above 20 minutes is a 1 point penalty. After a warning from me, *your* grace time may be shortened.

Worked Problems and Proofs. As the Notes are re-issued, problems, examples, lemmas, propositions, theorems, etc. that have been completed (proved or explained at the board) are marked $\Box$ at the end of the statement. The latest version will be available on Canvas.

Don’t stop here! Read on, next page.

1. Review of Naive Set Theory, Naive Logic, and Notation

Before we begin the study of topology, we must have a reasonably solid foundation in logic and set theory, including properties of functions. It is assumed that you are familiar with elementary logic and set theory, so this section should be review. This section is not meant to be exhaustive of the facts about sets and logic that may come up in the course. Some of the student work in this section is in the form of exercises. However, that does not mean that you should not do them. You are responsible for knowing the definitions and how to solve the exercises. And there are Problem for board work along the way. They will start to be presented on the first day of class by those smart enough to have read the Syllabus and Notes at least through Exercise 1.9.

1.1. Logic.

Logic begins with *statements*, which we won't define, except that they must be capable of being either true or false, but not both.

Remark 1.1. Note that "iff" is shorthand for "if, and only if."

Definition 1.2. Let p and q denote statements. The logical connectives, notation, and truth conditions are as follows:

(1) *Not* p, symbolized $\neg p$, is true iff p is false.
(2) *If* p*, then* q, symbolized $p \Rightarrow q$, is true, iff it is not the case that p is true and q is false. (Also called "p implies q.")
(3) p *iff* q, symbolized $p \Leftrightarrow q$, is true iff p and q have the same truth value. (Also called "p is equivalent to q.")
(4) p *and* q, symbolized $p \wedge q$, is true iff both p and q are true. (Also called the "conjunction" of p and q.)
(5) p *or* q, symbolized $p \vee q$ is true, iff at least one of p and q is true. (Also called the "disjunction" of p and q.)

Remark 1.3. I claim that $p \Rightarrow (q \Rightarrow p)$ is always true. For, to make it false p would have to be true and $q \Rightarrow p$ would have to be false. But to make $q \Rightarrow p$ false, q would have to be true and p false. But it is impossible for p to be both true and false. Hence, $p \Rightarrow (q \Rightarrow p)$ is always true.

Problem 1.4. *Which of the following statements are always true? The true ones we will call theorems of logic and use them in formulating proofs.*

(1) $p \Leftrightarrow q$ *is true iff* $(p \Rightarrow q) \wedge (q \Rightarrow p)$ *is true.*
(2) $p \Rightarrow q$ *is true iff* $\neg p \Rightarrow \neg q$ *is true.*
(3) $p \Rightarrow q$ *is true iff* $q \Rightarrow p$ *is true.*
(4) $p \Rightarrow q$ *is true iff* $\neg q \Rightarrow \neg p$ *is true.*
(5) $(p \wedge q) \Rightarrow r$ *is true iff* $p \Rightarrow r$ *and* $q \Rightarrow r$ *are both true.*
(6) $(p \vee q) \Rightarrow r$ *is true iff either* $p \Rightarrow r$ *or* $q \Rightarrow r$ *is true.*
(7) $(p \vee q) \Rightarrow r$ *is true iff* $p \Rightarrow r$ *and* $q \Rightarrow r$ *are both true.*

Exercise 1.5. *Form the denial of each of the following statements. Aim for "simplest form."*

(1) $p \Rightarrow q$.
(2) $p \Leftrightarrow q$.
(3) $p \wedge q$.
(4) $p \vee q$.

1.2. Set Theory.

We won't define what a set is either, but they have to be capable of having members.

Remark 1.6. We use the symbol $:=$ as shorthand for "is defined to be equal to" or "means the same as," whichever is appropriate.

Definition 1.7. Let A, B, C denote sets and let x, y, and z denote elements which may be members of a set. Set theoretic statements, connectives, and notation are defined as follows:

(1) *x is a member of A*, symbolized $x \in A$, is true iff x is an element in the set A. (We also use $A \ni x$ to mean $x \in A$.)
(2) $x \notin A := \neg(x \in A)$.
(3) *Sets A and B are equal*, symbolized $A = B$, is true iff A and B contain the same members.
(4) *A is a subset of B*, symbolized $A \subset B$, is true iff every member of A is also a member of B. (Equivalently, $A \subset B$ is false, symbolized $A \not\subset B$, iff there is a member of A which is not a member of B.) (We also use $B \supset A$ to mean $A \subset B$.)
(5) *The union of the sets A and B*, symbolized $A \cup B$, is the set $A \cup B := \{x \mid x \in A \vee x \in B\}$.
(6) *The intersection of the sets A and B*, symbolized $A \cap B$, is the set $A \cap B := \{x \mid x \in A \wedge x \in B\}$.
(7) *The difference of the sets A and B*, symbolized $A \setminus B$, is the set $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$. We call $A \setminus B$ the *complement* of B (in A).
(8) The empty set is the set with no members and is symbolized $\emptyset$.

Problem 1.8. *Which of the following set theoretic statements are true? If a statement is false, can you find a "related" statement which is true?*

(1) $A = B \Leftrightarrow (A \subset B) \wedge (B \subset A)$.
(2) $(A \subset B \wedge B \subset C) \Rightarrow A \subset C$.
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(4) *Can you switch the roles of $\cap$ and $\cup$ in the above statement?*
(5) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.
(6) *Can you replace $\cap$ by $\cup$ in the above statement?*
(7) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
(8) *Can you switch the roles of $\cap$ and $\cup$ in the above statement?*

Exercise 1.9. *Form the denial of each of the following statements. Aim for "simplest form."*

(1) $A \subset B$.
(2) $A = B$.
(3) $x \in A \cap B$.
(4) $x \in A \cup B$.

1.3. **More Logic: Quantifiers.** "Naive" set theory does not develop set theory axiomatically, and hence is subject to subtle contradictions (called paradoxes). It is unlikely that we will run into a paradox unless we try. We do not carefully distinguish between syntax (the study of well-formed formulas and of what is provable) and semantics (the study of truth). In particular, we assume that "the provable" and "the always true" coincide. These paradoxes and confusions can not arise until we introduce quantifiers. Once quantifiers are defined, and we maintain naiveté, we are subject to paradox and confusion, but don't let it worry you in this course.

Definition 1.10. Let $P(x)$ denote a statement about the individual x, usually in some set U of individuals. (Logicians sometimes call P a "predicate" since in the middle ages one would say "x is P" for $P(x)$; for example "x is mortal.") The quantifiers *for all* and *for some* have the following truth conditions:

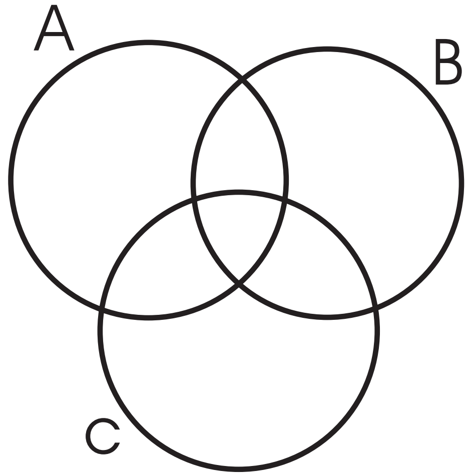

FIGURE 1. Venn diagram for three sets: a way of discovering truths.

(1) *For all $x \in U$, $P(x)$*, symbolized $\forall x \in U, P(x)$, is true iff, for every individual x in the set U, P holds true for that individual x.
(2) *For some $x \in U$, $P(x)$*, symbolized $\exists x \in U, P(x)$, is true iff there is at least one individual $y \in U$ for which $P(y)$ holds true.

Remark 1.11. Note that if $U = \emptyset$, then $\forall x \in U, P(x)$ is "vacuously" true, and $\exists x \in U, P(x)$ is necessarily false.

Exercise 1.12. *What must be the case if both $\forall x \in U, P(x)$ and $\forall x \in U, \neg P(x)$ are true?*

Remark 1.13. One can also have compound predicates; for example, $R(x_1, x_2 \dots, x_n)$ and logical statements with individual variables in them. In that case, each variable $x_1, x_2 \dots, x_n$ should have a quantifier. For examples, see items (7) and (8), below.

Problem 1.14. *Form the denial of statements (1)-(5), (7), (8) following. Also answer (6) and (9).*

(1) $\forall x \in U, P(x)$.
(2) $\exists y \in V, Q(y)$.
(3) $\forall x \in U, (P(x) \Rightarrow Q(x))$.
(4) $\forall x \in U, \exists y \in V, (P(x) \Rightarrow Q(y))$.
(5) $(\forall x \in U, (P(x)) \Rightarrow (\exists y \in V, Q(y))$.
(6) *Explain the difference between the previous two statements.*
(7) $\forall \varepsilon > 0, \forall x \in U, \exists \delta > 0, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
(8) $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in U, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
(9) *Explain the difference between the previous two statements.*

Remark 1.15. We will use the term "collection" synonymous with the term "set" in order to avoid phrases such as "set of sets... ." For clarity in symbolizing "levels" of sets, we will use the following conventions:

(1) Lower case Roman letters $a, b, c, \dots$ denote elements of sets that are typically not themselves sets.
(2) Upper case Roman letters $A, B, C, \dots$ denote sets.
(3) Caligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ denote collections of sets.
(4) Lower case Greek letters $\alpha, \beta, \gamma, \dots$ sometimes denote indices, when we want to index a collection of sets in order to distinguish the members. But $i, j, k, \dots$ may also denote indices, particular if the indices are natural numbers or integers.

(5) Upper case Greek letters are used as needed (sometimes as a set of indices.)

We can use quantifiers to extend our set theoretic connectives and statements to arbitrary collections of sets.

Definition 1.16. Let $\mathcal{A}$ and $\mathcal{B}$ denote collections of sets. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ denote an indexed collection of sets. Set theoretic connectives and statements may be further extended as below. (This is only a sample.)

(1) *The union of the collection* $\mathcal{A}$, symbolized $\bigcup\mathcal{A}$, is the set $\bigcup\mathcal{A} := \{x \mid \exists A \in \mathcal{A}, x \in A\}$.
(2) *The intersection of the collection* $\mathcal{A}$, symbolized $\bigcap\mathcal{A}$, is the set $\bigcap\mathcal{A} := \{x \mid \forall A \in \mathcal{A}, x \in A\}$.
(3) *The union of the indexed collection* $\{A_\alpha\}_{\alpha\in\Gamma}$, symbolized $\bigcup_{\alpha\in\Gamma} A_\alpha$, is the set $\bigcup_{\alpha\in\Gamma} A_\alpha := \{x \mid \exists\alpha \in \Gamma, x \in A_\alpha\}$.
(4) *The intersection of the indexed collection* $\{A_\alpha\}_{\alpha\in\Gamma}$, symbolized $\bigcap_{\alpha\in\Gamma} A_\alpha$, is the set $\bigcap_{\alpha\in\Gamma} A_\alpha := \{x \mid \forall\alpha \in \Gamma, x \in A_\alpha\}$.

In case the indexed collection $\{A_i\}_{i\in\Gamma}$ is indexed by a finite or countably infinite set, for example $\Gamma = \{1, 2, \ldots, n\}$ or $\Gamma = \mathbb{Z}^+$, we usually write unions (likewise, intersections) as $\bigcup_{i=1}^n A_i$ or $\bigcup_{i=1}^\infty A_i$.

Problem 1.17. *Which of the following set theoretic statements are true? If a statement is false, can you find a "related" statement which is true?*

(1) $A \cap (\bigcup\mathcal{B}) = \bigcup_{B\in\mathcal{B}}(A \cap B)$. *(Note that* $\mathcal{B}$ *is being used as its own index set.)*
(2) $A \setminus (\bigcup\mathcal{B}) = \bigcap_{B\in\mathcal{B}}(A \setminus B)$.
(3) *Can you switch the roles of* $\cap$ *and* $\cup$ *in the above statements?*
(4) *Suppose* $\mathcal{A} = \emptyset$. *What is* $\cup\mathcal{A}$*?*
(5) *Suppose* $\mathcal{A} = \emptyset$. *What is* $\cap\mathcal{A}$*?*

1.4. **Proofs.** You have nearly all had at least one semester of Advanced Calculus or another course in which you constructed proofs, so it is assumed that you know what a proof is. I remind you that there are generally three methods of proof: (1) direct, (2) by contraposition, and (3) indirect (by contradiction). There is also the technique of proof by mathematical induction. A counter-example is essentially a proof that a statement is false, but it requires existence; that is, a counter-example must be specific and name individuals and sets.

Exercise 1.18. *Explain how you might go about proving (or disproving, if false) each statement below. (If you skip this you will be sorry!)*

(1) $\forall x \in U, (P(x) \Rightarrow Q(x))$.
(2) $\forall x \in U, P(x) \Rightarrow \exists x \in U, P(x)$.
(3) $\exists x \in U, P(x) \Rightarrow \forall x \in U, P(x)$.
(4) $\forall x \in U, (P(x) \Leftrightarrow Q(x))$.
(5) $\forall x \in U, \exists y \in V, (P(x) \Rightarrow Q(y))$.
(6) $(\forall x \in U, (P(x)) \Rightarrow (\exists y \in V, Q(y))$.
(7) $\forall\varepsilon > 0, \forall x \in U, \exists\delta > 0, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.
(8) $\forall\varepsilon > 0, \exists\delta > 0, \forall x \in U, \forall y \in U, (d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon)$.

1.5. **Functions and Relations.**

Definition 1.19. The *Castesian product* of two sets A and B is the set of ordered pairs defined as follows: $A \times B := \{(a, b) \mid a \in A \vee b \in B\}$. A *relation* between A and B is any subset of the Cartesian product.

Definition 1.20. Let A and B be sets. A *function* f, symbolized $f : A \to B$, is an unambiguous rule that assigns to each element of the set A an element of the set B.

As such, a functiuon can be regarded as particular type of relation in $A \times B$: one in which every element of A must appear in some ordered pair, and no two ordered pairs in f can have the same first element. We also use the following terminology.

Set A is called the *domain* of f and set B is called the *range* of f. For each $a \in A$, we use $f(a)$ to denote the member of B assigned to a. Let $C \subset A$. We use the notation $f(C) := \{b \in B \mid \exists a \in C, f(a) = b\}$, and call $f(C)$ the *image* of C.

The function f is *onto* (equivalently, *surjective*) iff $f(A) = B$. The function f is *one-to-one* (equivalently, *injective*) iff $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. Let $D \subset B$. It is convenient to use the notation $f^{-1}(D) := \{a \in A \mid \exists d \in D, f(a) = d\}$. We call $f^{-1}(D)$ the *pre-image* of D. A function $f : A \to B$ is a *one-to-one correspondence* (equivalently, *bijective*) from A to B iff f is both onto and one-to-one. Be warned that f^{-1} is not always a function (from B to A).

Exercise 1.21. *Suppose $f : A \to B$ is a function and $C \subset A$.*

(1) *Why do you suppose we do not generally call f^{-1} the* inverse *of f?*
(2) *Suppose that $f : A \to B$ is a one-to-one correspondence? Show there is a one-to-one correspondence from B to A. (We call this one-to-one correspondence the* inverse *of f and symbolize it $f^{-1} : B \to A$.)*
(3) *What can you say about the statement $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$?*
(4) *Is $f : A \to f(A)$ a function? What kind now?*
(5) *Is $f|_C : C \to B$ a function? (Here $f|_C$ is the function defined by $f|_C(a) := f(a)$ for all $a \in C$. We call $f|_C$ the* restriction *of f to C, whether the range is $f(C)$ or B.)*

Problem 1.22. *Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$ are functions.*

(1) *Show that $g \circ f : A \to C$ defined by $(g \circ f)(a) := g(f(a))$ is a function from A to C. (We call $g \circ f$ the* composition *of f and g.)*
(2) *Show that f and g onto (respectively, one-to-one) implies $g \circ f$ is onto (respectively, one-to-one).*
(3) *Show by example that $g \circ f$ is not necessarily the same as $f \circ g$, even when both are defined.*

Problem 1.23. *Let A and B be subsets of X, and $f : X \to Y$ a function. Determine the relationship between the following pairs:*

(1) *$f(A \cap B)$ and $f(A) \cap f(B)$.*
(2) *$f(A \cup B)$ and $f(A) \cup f(B)$.*
(3) *$f(A \setminus B)$ and $f(A) \setminus f(B)$.*
(4) *$A \subset B$ and $f(A) \subset f(B)$.*

Problem 1.24. *Let A and B be subsets of Y, and $f : X \to Y$ a function. Determine the relationship between the following pairs of sets.*

(1) *$f^{-1}(A \cap B)$ and $f^{-1}(A) \cap f^{-1}(B)$.*
(2) *$f^{-1}(A \cup B)$ and $f^{-1}(A) \cup f^{-1}(B)$.*
(3) *$f^{-1}(A \setminus B)$ and $f^{-1}(A) \setminus f^{-1}(B)$.*
(4) *$A \subset B$ and $f^{-1}(A) \subset f^{-1}(B)$.*

Exercise 1.25. *Suppose $f : A \to B$ is an onto function.*

(1) *Describe how you might go about defining, using f, a one-to-one correspondence with B as range. (Must the domain be a subset of A?)*

(2) *Describe how you might go about defining a function* $g : B \to A$ *so that* $g \circ f = id_A$. *(*id_A *denotes the identity function of* A *to itself.)*
(3) *In the above, can you define* g *so that* $f \circ g$ *is the identity on* B*?*

E-mail address, John C. Mayer: `jcmayer@uab.edu`

(John C. Mayer) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170