SECTION 5

NUMERICAL METHODS
INCREMENTAL MATRIX INVERSION APPROACH FOR RADIAL BASIS FUNCTION MESH DEFORM

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ABSTRACT
An important step in the analysis of fluid-structure interaction problems using a computational approach is the fluid mesh deformation based on the structural deflections. A new methodology is presented for the deformation of volume meshes with arbitrary topology using a radial basis functions (RBF) based interpolation. A greedy algorithm is used to select a small subset of the surface nodes to accelerate the interpolation. In this study, an incremental approach is used to compute the inverse of the RBF evaluation matrix by utilizing the matrix system from the previous greedy algorithm's iteration. Results are presented from an accuracy study of the current approach using a specified analytical deformation on a surface. Also, computational results are presented for specified deflections of a rectangular super critical wing. These simulations showed that the incremental approach takes significantly less CPU time than a traditional matrix solver.

NOMENCLATURE

- $d$ = displacement vector
- $h$ = mesh refinement
- $n$ = number of nodes
- $r$ = support radius
- $S$ = interpolation function
- $X$ = node coordinate vector
- $\|x\| = \text{Euclidean distance}$
- $\alpha$ = weight coefficient vector
- $\phi$ = radial basis function

Subscripts:
- $b$ = boundary nodes
- $s$ = selected boundary node

INTRODUCTION
Fluid Structure Interaction (FSI) and control surface deflections are examples of dynamic computational simulations that involve moving/deforming meshes. A critical step in the analysis of this class of problems is a successful mesh deformation due to structural deflections. For an accurate simulation, the fluid volume mesh needs to move conformal to the structure without compromising the mesh quality. An efficient and reliable approach is needed for a successful analysis due to the repeatability of the mesh deformation and the large number of fluid cells. The main challenge of the moving mesh approach is to find an optimum technique that is suitable for different mesh topologies and physical situations. At the same time, it should preserve the quality of the mesh without increasing the computational cost. The objective of this study is to investigate the feasibility of improving the performance of the radial basis function (RBF) based interpolation technique with a greedy algorithm to deform large-scale generalized meshes by using an incremental approach.

Strategies for deforming the fluid mesh conforming to the deformation of the structure can be divided into two basic classes: physical analogy and interpolation. A brief literature review of each class is discussed in the following section.

MESH DEFORMATION USING PHYSICAL ANALOGY
The physical analogy approach treats the fluid mesh deformation problem as a physical process that can be modeled using numerical methods. One of the popular methods in this class is the tension spring analogy proposed by Batina [1]. In this approach, each edge of the mesh is replaced by a tension spring, with the spring stiffness inversely proportional to the edge length. The main disadvantage of this approach is the mesh crossing problem. One of the improvements to this approach is to consider the fluid mesh as a network of torsional...
springs added at the nodes to prevent cell collapse and to allow cell rotation [2]. The main drawback of these methods is that they involve solving large systems of equations, implying a higher computational cost. Besides, physical analogy based methods require grid connectivity information, which results in more storage requirements and difficulties in parallelization.

A later advancement under the physical analogy approach is to interpret the mesh as a continuous elastic medium by using the multidimensional linear elasticity analogy [3]. The modulus of elasticity is chosen to be inversely proportional to the distance from the deforming boundaries or to the cell volume. In this approach, each displacement component of a mesh movement is governed by a partial differential equation, such as Laplacian equation [4].

**MESH DEFORMATION USING INTERPOLATION ANALOGY**

In this approach, prescribed boundary point displacements are transferred to the fluid mesh using an interpolation function. In general, these schemes do not require grid connectivity information. Therefore, this approach can be used to deform arbitrary mesh types that contain general polyhedral elements or hanging nodes [5].

Recently, a novel interpolation based scheme has been developed by Luke [5]. In this scheme, the deformation of the volume mesh is viewed as a projection of the surface deformation into the volume. Using a tree-code optimization, the algorithm cost is demonstrated to be $O(n \log(n))$, where $n$ is the total number of nodes in the simulation, with mesh quality that is competitive to the radial basis function (RBF) scheme.

A technique that is based on a two-pronged approach, where the viscous layers of nodes are deformed rigidly and the outer region is deformed with two different interpolation techniques, was developed McDaniel and Morton [6]. Their results showed that the best performing scheme was based on a semi-rigid connection to the owner surface nodes, defined as part of the mesh parsing. They used the last layer of the viscous region as the deforming boundary surface for the outer region deformation.

The radial basis function interpolation method, such as the method developed by Boer et al. [7],[8], is one of the promising interpolation schemes. Although RBFs are commonly used to interpolate scattered data, they can also be used as interpolation functions to transfer the prescribed displacements at the boundaries of the structural mesh to the fluid mesh. This scheme has been reported to produce high-quality meshes with reasonable orthogonality preservation near deforming boundaries [8]. Other advantages of RBF include: 1) Avoiding the need for mesh connectivity information, 2) solving linear systems of equations, and 3) the size of the linear system of equation is proportional to the number of boundary nodes, not all fluid nodes. Moreover, many studies have investigated different techniques for improving RBF’s interpolation based mesh deformation. The most influential study was made by Rendall and Allen [9]. They proposed the use of a greedy algorithm along with RBF interpolation. More details about this technique will be discussed in the following sections. Another study, which builds on Rendall and Allen [9], is the work by Sheng and Allen [10], in which they put forward specific criteria for selecting the nodes involved in the interpolation.

**RBF FORMULATION**

The interpolation function, $S$, describing the displacement in the whole domain can be approximated by a sum of basis functions as,

$$S(X) = \sum_{j=1}^{n_b} \alpha_j \phi \left( \frac{\|X - X_{b_j}\|}{r} \right) + P(X)$$

(1)

where $X_{b_j} = (x_{b_j}, y_{b_j}, z_{b_j})$ are the boundary nodes in which the deformations are known, and these are called the centers for RBF. $P$ is a polynomial, $n_b$ is the number of boundary nodes, and $\phi$ is the selected basis function with respect to the Euclidean distance $\|x\|$. The coefficients $\alpha_j$ and the polynomial $P$ are determined by the interpolation conditions

$$S(X_{b_j}) = d_{b_j}$$

(2)

$$\sum_{j=1}^{n_b} \alpha_j = \sum_{j=1}^{n_b} \alpha_j x_j = \sum_{j=1}^{n_b} \alpha_j y_j = \sum_{j=1}^{n_b} \alpha_j z_j = 0$$

(3)

The values for the coefficients $\alpha_j$ and the linear polynomial coefficient can be obtained by solving the system

$$\begin{bmatrix} \phi_{b,b} & P \end{bmatrix} \begin{bmatrix} \alpha \\ p \end{bmatrix} = \begin{bmatrix} d_b \\ 0 \end{bmatrix}$$

(4)

where $\alpha$ is a vector containing the coefficients $\alpha_j$, $\beta$ is a vector containing the coefficients of the linear polynomial $P$, $\phi_{b,b}$ is an $n_b \times n_b$ matrix containing the evaluation of the basis function $\phi_{b,b,j} = \phi \left( \frac{\|X_{b_j} - X_b\|}{r} \right)$, and $p$ is an $n_b \times 4$ matrix with row $j$ given by $[1 \ x_j \ y_j \ z_j]$ [7],[8].

Since the polynomial $P$ was concluded in previous studies to not have a significant influence on the quality of the deformed mesh, the polynomial $P$ was omitted in this study [12]. In this case, the system of equations in (4) will be simplified as the following

$$\phi_{b,b} [\alpha] = [d_b]$$

(5)

Generally, RBFs can be divided into two categories 1) functions with compact support, and 2) functions with global support. The main difference between the two categories is that the function with compact support is scaled with a support radius ($r$) to control the extent of influence of the basis function. Functions with global support cover the whole
interpolation space, which leads to dense matrix systems. The following table lists some RBFs for both these types \([7,8]\).

<table>
<thead>
<tr>
<th>No.</th>
<th>Function (\phi)</th>
<th>Type</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1 - \xi)^2)</td>
<td>C.S.</td>
<td>(\xi = x/r)</td>
</tr>
<tr>
<td>2</td>
<td>((1 - \xi)^4(4\xi + 1))</td>
<td>C.S.</td>
<td>(\xi = x/r)</td>
</tr>
<tr>
<td>3</td>
<td>((1 - \xi)^6(35/3\xi^2 + 6\xi + 1))</td>
<td>C.S.</td>
<td>(\xi = x/r)</td>
</tr>
<tr>
<td>4</td>
<td>(e^{-x^2})</td>
<td>G.S.</td>
<td>Gaussian</td>
</tr>
<tr>
<td>5</td>
<td>(x^2 \log(x))</td>
<td>G.S.</td>
<td>Thin Plate Spline</td>
</tr>
<tr>
<td>6</td>
<td>(1 + x^2)</td>
<td>G.S.</td>
<td>Quadric</td>
</tr>
</tbody>
</table>

where C.S.: compact support and G.S.: global support.

Functions with compact support are forced to satisfy the following condition

\[
\phi_{b,b} = \begin{cases} \phi(\|X_b - X_b\|) & \text{if } 0 \leq \xi \leq 1 \\
0 & \text{if } \xi > 1 
\end{cases}
\]  

\[ (6) \]

**DIRECT RBF SCHEME LIMITATIONS AND IMPROVEMENTS**

The radial basis function interpolation method produces high-quality meshes with good orthogonality preservation near deforming boundaries. On the other hand, in its most straightforward implementation, it is computationally expensive to use for large 3-D problems. A direct solution of such systems requires \(O(n_b^3)\) operations and \(O(n_b^2)\) memory usage, which becomes impractical for problems with more than a few thousands grid points. Great progress has been made in recent years towards alleviating this computational burden.

An approximation approach for RBF mesh deformation has been suggested by Rendall and Allen \([9]\). In this approach, the RBF is applied using a coarsened subset of the surface mesh. Displacements of the omitted surface nodes are calculated using the interpolation method, and the error is calculated as the difference between the interpolated values and the actual displacement. A greedy algorithm is used to add points that have the largest error. Rendall and Allen reported that this algorithm improves the performance of the RBF method by approximately two orders of magnitude. The use of the greedy algorithm reduces the cost remarkably without any noticeable accuracy loss. This study takes advantage of solving similar systems of equations within each iteration by using an incremental matrix inversion approach \([13]\) to reduce the CPU time by avoiding the use of any expensive linear system solver such as LU decomposition. Since the proposed method is not affected by the matrix’s condition number, it does not require any pre-conditioning. The procedure of selecting the interpolation centers using the greedy algorithm and the use of the incremental approach is described in the following section.

**GREEDY ALGORITHM WITH INCREMENTAL APPROACH**

For large-scale unstructured-grid problems, the number of boundary nodes is usually of the order \(10^4\) to \(10^5\); therefore, calculating the interpolation weight coefficients based on all boundary nodes tends to be very costly. Moreover, if the mesh is required to be deformed at each time step, the direct RBF method cost will be impractical. It was reported in the literature that to achieve high mesh deformation accuracy, only a small subset of the total boundary nodes is sufficient \([9,10]\).

Centers selection is highly dependent on the displacement vector, shape of the geometry, and the support radius in the case of compact support RBFs. Therefore, there is no direct way to select these nodes in advance yet. Hence, a greedy algorithm is used to iteratively select these centers. Initially, two nodes are selected during the first iteration. For subsequent iterations, the selected centers are used to predict the deformation of the unselected centers. Based on this prediction the error is calculated at all points, which is the difference between the actual deformation and the predicted value of the deformation. The greedy algorithm is stopped if the maximum error is below a specified tolerance. Otherwise, the node with the maximum error is added to the selected subset, and the iteration continues.

Sheng and Allen \([10]\) investigated the possibility of selecting the centers based only on the problem geometry. Thus, the greedy algorithm is used only once, and the same subset of centers is used for all time steps for different deformations.

Assume the greedy algorithm starts with \(n\) selected centers. Equation (5) then becomes

\[
\begin{bmatrix}
\phi_{1,1} & \cdots & \phi_{1,n} \\
\vdots & \ddots & \vdots \\
\phi_{n,1} & \cdots & \phi_{n,n}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
= 
\begin{bmatrix}
d_1 \\
\vdots \\
d_n
\end{bmatrix}
\]  

or

\[
[\phi_n][\alpha_n] = [d_n]
\]

where \(\phi_{s,s} = \phi(\|X_{\text{selected}} - X_{\text{selected}}\|)\) and \(\phi_{u,s} = \phi(\|X_{\text{unselected}} - X_{\text{selected}}\|)\).

Then for the next iteration, the node with the largest error will be added to the list, and the system will become

\[
\begin{bmatrix}
\phi_{1,1} & \cdots & \phi_{1,n} & \phi_{1,n+1} \\
\vdots & \ddots & \vdots & \vdots \\
\phi_{n,1} & \cdots & \phi_{n,n} & \phi_{n,n+1} \\
\phi_{n+1,1} & \cdots & \phi_{n+1,n} & \phi_{n+1,n+1}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_n \\
\alpha_{n+1}
\end{bmatrix}
= 
\begin{bmatrix}
d_1 \\
\vdots \\
d_n \\
d_{n+1}
\end{bmatrix}
\]  

or

\[
[\phi_{n+1}][\alpha_{n+1}] = [d_{n+1}]
\]

Note that

\[
[\phi_{n+1}] = 
\begin{bmatrix}
\phi_n & \phi_{l,n+1} \\
\phi_{n+1,1} & \phi_{n+1,n+1}
\end{bmatrix}
\]

where \(i = [1, 2, \ldots, n]\).
For compact support RBFs, the matrix $\phi_{b,b}$ is symmetrical with one in the diagonal. Therefore, the system can be simplified as

$$
\begin{bmatrix}
\phi_n & \phi_{add} \\
\phi_{add}^T & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_n \\
\alpha_{n+1}
\end{bmatrix} =
\begin{bmatrix}
d_1 \\
\vdots \\
d_n \\
\vdots \\
d_{n+1}
\end{bmatrix}
$$

(12)

where $\phi_{add}$ is a vector that contains the RBF evaluations for the newly added node with respect to each selected center.

The most straightforward technique for solving the above system is by calculating the inverse of the coefficients matrix. Since the coefficient matrix consists of the matrix from the previous iteration with an addition of one row and one column, it is possible to use an incremental approach to compute the inverse using the inverse from the previous iteration. Details of the incremental approach are discussed below.

INCREMENTAL RBF COMPUTATION

The inverse of a general block matrix can be calculated as [14]

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1} = \begin{bmatrix}
(A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1}
\end{bmatrix}
$$

(13)

Applying this principle to the calculation of the inverse of the coefficient matrix in Equation (11) will result in,

$$
\begin{bmatrix}
\phi_n & \phi_{add} \\
\phi_{add}^T & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
(\phi_n - \phi_{add} \phi_{add}^T)^{-1} & \frac{1}{k} \phi_n^{-1} \phi_{add} \\
\frac{1}{k} \phi_{add} \phi_n^{-1} & \frac{1}{k}
\end{bmatrix}
$$

(14)

where $k = 1 - \phi_{add} \phi_n^{-1} \phi_{add}$

Since the matrix $\phi_n$ is symmetrical, the previous equation can be simplified as

$$
[\phi_{n+1}]^{-1} = \frac{1}{k} \left[
\begin{bmatrix}
\xi \phi_n^{-1} + \xi \xi^T & -\xi \\
-\xi^T & 1
\end{bmatrix}
\right]
$$

(15)

where $\xi = \phi_n^{-1} \phi_{add}$

This approach reduces the complexity required by traditional methods, e.g., Gauss elimination or LU decomposition, in this specific case from $O(n^3)$ [13] to $O(n^2)$ complexity [15]. Moreover, the complexity reduction for this step is more significant since it is repeated in every iteration until the convergence is achieved. However, it must be noted here that during the first greedy algorithm's iteration, the inverse of the matrix must be computed using any traditional method. At this specific iteration, the system is very small in size, typically 2x2, which is very feasible to solve.

RESULTS AND DISCUSSION

Two-dimensional test functions

Four analytical test functions were chosen to deform a structured mesh in order to investigate the accuracy and efficiency of the presented method. To analyze the influence of the total number of nodes on the algorithm performance, a mesh refinement study is conducted. Three different structured meshes used for this study are shown in Figure 1. The x- and y-coordinates of the computational domain varies from -1.5 to 1.5. The z-coordinate of the computational domain is set to zero originally, and it is specified as the analytic functions after the deformation. The specified deformations of the mesh for this study are listed below.

$$
F_1(x, y) = (1 + 9x^2 + 16y^2)^{-1}
$$

(16)

$$
F_2(x, y) = 1 - \left((x^2 + y^2)/2\right)^{1/2}
$$

(17)

$$
F_3(x, y) = 1.5xe^{-x^2-y^2}
$$

(18)

$$
F_4(x, y) = (1.25 + \cos(3.4y)) / (6 + 6(3x - 1)^2)
$$

(19)

Figure 1. Surface mesh with three different element sizes (h) 0.1, 0.05, and 0.025 respectively

The displacement at each node is pre-calculated using the above functions with two random nodes selected as initial centers for the RBFs. The greedy algorithm uses the initial centers to calculate the RBF weight coefficients and to evaluate the displacement for all the remaining nodes. The error is calculated as the difference between the pre-calculated displacement and the evaluated displacement to check the stopping criteria for the greedy algorithm. If the maximum error is greater than the specified tolerance, the node with the maximum error is added to the list of centers for RBF. If the maximum error is less than the specified tolerance, the greedy algorithm stops the iteration and proceeds with deformation of the fluid mesh using the selected RBF centers. The performance of the proposed technique was compared against the use of the conventional greedy algorithm and against the use of the direct RBF method. Table 2 shows the results of this comparison, which demonstrate the significant CPU time reduction achieved by using the matrix inversion incremental approach. Moreover, the conventional greedy algorithm showed better performance than the direct method for all shape functions except for shape function F2. Shape function F2 requires picking relatively a higher number of centers, to achieve the predefined tolerance, compared to other shape functions. This leads the system to perform additional iterations of the greedy algorithm, which consequently leads to a longer CPU time.
The number of selected centers and the total number of nodes on the surface for three meshes with different resolution and for four different deformations are compared in Figure 2. In these calculations, the error tolerance for stopping the greedy algorithm was $1.0 \times 10^{-3}$, and a support radius of $1.5$ was used. It is clear that the mesh refinement did not cause the number of selected centers to increase. The number of centers required for each of the analytical deformation remains almost the same irrespective of the number of surface nodes. However, the number of selected nodes varies for different analytical functions used for the deformation.

Table 2. CPU Time in seconds for direct RBF, RBF with conventional greedy algorithm, and RBF with incremental greedy algorithm for element size $(h) = 0.1$ and tolerance $= 5 \times 10^{-4}$

<table>
<thead>
<tr>
<th>Shape Function</th>
<th>Direct RBF</th>
<th>Conventional Greedy</th>
<th>Incremental Greedy</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>9.33</td>
<td>1.52</td>
<td>0.071</td>
</tr>
<tr>
<td>F2</td>
<td>10.13</td>
<td>12.49</td>
<td>0.223</td>
</tr>
<tr>
<td>F3</td>
<td>9.55</td>
<td>0.274</td>
<td>0.038</td>
</tr>
<tr>
<td>F4</td>
<td>9.57</td>
<td>1.63</td>
<td>0.072</td>
</tr>
</tbody>
</table>

Figure 2. Total number of nodes versus selected number of centers for the three different mesh refinements 0.1, 0.05, and 0.025

The specified four analytic deformations used for this study and the selected RBF centers for the interpolation for three different meshes are shown in Figure 3. In this figure the color contour is specified based on the deformation values. It can be seen that the patterns of the selected RBF centers are the same for all different mesh resolutions.

Figure 3. Original surface mesh and selected centers (top view) for all three refinements

Figure 4. Deformed mesh colored by absolute error

In order to analyze the effect of error tolerance on the number of selected centers and the CPU time required for the selection of appropriate centers, the number of selected centers and CPU time requirements were calculated for error tolerances of $1 \times 10^{-3}$, $5 \times 10^{-4}$, and $1 \times 10^{-4}$. The fine mesh ($h = 0.025$) from the mesh refinement study is used for this analysis. The finer the specified tolerance, the higher the number of centers picked and longer the CPU time consumed. Figure 5 shows the variation of CPU time and number of selected centers versus the specified tolerance. It is noticeable from the chart that the number of selected centers and the CPU time for the greedy algorithm increase continuously with the decrease in error tolerance.

To determine the dependency of the total number of nodes $(N)$ on the interpolation error, the Root Mean Square (RMS) error is plotted against the number of nodes using a logarithmic scale in Figure 6. It can be seen from the figure that all different analytical functions for deformation showed an approximate slope of 0.4 or less, indicating that the total number of nodes has an insignificant effect on the interpolation error.
The computational time savings for the presented incremental matrix inversion algorithm and traditional RBF interpolation with greedy algorithm are compared using a Rectangular Supercritical Wing (RSW) test case. This RSW test case was used at the Aeroelastic Prediction Workshop sponsored by the Structural Dynamics Technical Committee, American Institute of Aeronautics and Astronautics (AIAA) [17],[18]. This wing has a span of 48 inches and root chord length of 24 inches. The mesh used for this analysis is one of the meshes provided for the Aeroelastic Prediction Workshop and is available for download from the workshop website†. A fully tetrahedral mesh was selected for this test case as shown in Figure 7. The fluid domain has the dimensions of 4824x2400x4800. The mesh consists of 17,453,792 cells, 2,944,006 nodes, and 56,272 boundary nodes. A direct RBF interpolation technique will require operations of the order of 2,944,006 x 56,272 to evaluate the deformation for all interior nodes. In addition, this process will be repeated for each time step, which will make it impractical for FSI analysis.

However, it can be noted from the results that the time required for the greedy algorithm with incremental approach is significantly smaller than that for the greedy algorithm with a support radius of 1200 were used. The support radius was calculated as half the distance between the furthest fluid node and any boundary node.

Pictorial views of the mesh before and after applying the deformations are shown in Figure 8. The quality of the deformed meshes was examined and no negative volume cells were found without any noticeable change in the mesh skewness before and after the deformation. The computational costs and number of selected centers for the mesh deformations for these three cases are compared in Table 3. This Table also includes the CPU timing for a conventional method, namely LU decomposition method, for matrix inversion as well as for the incremental approaches. The reason to use the LU decomposition method as the base method for comparison instead of using an iterative method is that the matrix system was highly ill-conditioned. It is clear that the number of selected centers and accordingly the corresponding computational time increases for larger deformations. However, it can be noted from the results that the time required for the greedy algorithm with incremental approach is significantly smaller than that for the greedy algorithm with a conventional matrix solver. Moreover, the superiority of the incremental approach increases for larger deformations and accordingly for larger meshes, as illustrated by the increase of the computational time saving with the increase of the deformation angle.

Table 3. Greedy algorithm’s CPU time in seconds for different deformation angles (C: conventional method and I: incremental approach)

<table>
<thead>
<tr>
<th>Deform Angle</th>
<th>Greedy Algorithm (C / I)</th>
<th>Evaluate Interior Nodes</th>
<th>Total (C / I)</th>
<th>Total Saving</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0°</td>
<td>11.62 / 8.24</td>
<td>6.85</td>
<td>18.5 / 15.1</td>
<td>18.3%</td>
</tr>
<tr>
<td>3.5°</td>
<td>58.86 / 24.75</td>
<td>13.96</td>
<td>72.8 / 38.7</td>
<td>46.8%</td>
</tr>
<tr>
<td>5.0°</td>
<td>162.65 / 35.34</td>
<td>17.72</td>
<td>180.4 / 53.1</td>
<td>70.6%</td>
</tr>
</tbody>
</table>
CONCLUSION

An efficient mesh deformation technique using radial basis functions with an incremental greedy algorithm for FSI simulations is presented in this paper. An incremental approach was proposed for the inversion of the matrix system during each greedy iteration. The use of the incremental approach reduced the computational complexity of the matrix inversion step within each greedy algorithm's iteration from $O(n^3)$ to $O(n^2)$. Benchmark test cases with four different analytic deformations are used to evaluate the performance of the presented approach. The results from the numerical experiments showed that the number of centers required to perform the interpolation is independent of the total number of nodes and mainly depends on the deformation. This makes the technique optimal for large-scale FSI problems where the meshes are very fine. Moreover, the algorithm's order of accuracy is also independent of the total number of nodes. Results are also presented for mesh formations for deflections of a rectangular supercritical wing. These simulations showed the incremental approach takes significantly less CPU time as compared to the traditional matrix solver. The present results show that improvement in CPU time saving increases as the number of selected centers for RBF increases.

REFERENCES

ABSTRACT
Piezoelectricity is an electric charge that some materials generate in response to applied mechanical stress. If the material is not short-circuited, the applied charge induces a voltage across the material. This technology, together with innovative mechanical coupling designs, can form the basis for harvesting energy from mechanical motion. In this study Flex PDE is used for FEM modeling of a unimorph piezoelectric ring under internal distributed sinusoidal pressure. The transient response of PZT-4 and PZT-5A materials has been studied, and results are compared with previous studies. The simulation predicts that electric charge will be produced by piezoelectric radial deformation. Hence, a piezoelectric ring can be used in an energy harvesting system.

INTRODUCTION
The term “energy harvesting” refers to the generation of energy from sources such as ambient temperature, vibration or air flow. Vibration-based energy harvesting has attracted the attention of researchers over the last decade. The ultimate goal of research in this field is to power small electronic components by using the vibration energy that is available in their environment. It can be seen that piezoelectric transduction among the basic transduction mechanisms has received the most attention which can be used for vibration-to-electricity conversion. Piezoelectric materials, based on their large power densities and ease of application, are preferred in energy harvesting. Ding et al. [1] investigate the transient responses of piezoelectric hollow cylinders for axisymmetric plane strain problems. Williams et al. [2] presented the first idea of vibration-to-electricity conversion. They described the basic transduction mechanisms that can be used for this purpose and provided a lumped-parameter base excitation model to simulate the electrical power output for electromagnetic energy harvesting. Beeby et al. [3] studied vibration energy harvesting for wireless, self-powered Microsystems. They presented the characteristic equations for inertial-based generators, along with the specific damping equations. They described three main transduction mechanisms employed to extract energy from the system and a comprehensive review of existing piezoelectric generators, including impact coupled, resonant and human-based devices. Cook-Chennault et al. [4] provided an overview of strategies for powering MEMS via non-regenerative and regenerative power supplies. Kamel et al. [5] considered a physical model for predicting the generated electric power from piezoelectric harvesting devices. Their model is based on estimating the total charge generated on a piezoelectric material when it is subjected to mechanical strain as a result of bending at the fundamental resonance frequency. Renaud et al. [6] proposed an alternative design based on the impact of a moving mass on piezoelectric bending structures. Their model is considered to determine the parameters influencing the device performance in terms of energy harvesting. Majidi et al. [7] applied an array of vertically aligned zinc oxide (ZnO) nanoribbons to harvest nanoscale vibrational energy. Wang et al. [8] used a curved beam in the cavity of a sonic crystal to harvest acoustic energy. Zurkinden et al. [9] investigated the harvesting mechanism of ocean surface wave energy using PVDF films. Kuehne et al. [10] studied a piezoelectric harvesting micro generator for a tire pressure monitoring wireless sensor node. They investigated the fluid-structure interaction of the energy harvesting MEMS generator with the surrounding gas. Pallapa et al. [11] calculated the performance of piezoelectric micro-power generators numerically using COMSOL, ANSYS, and Coventor. Erturk et al. [12] presented the exact analytical solution of a cantilevered piezoelectric energy harvester with Euler–Bernoulli beam assumptions. Simple expressions for the coupled mechanical response, voltage, current, and power outputs are also presented for excitations around the modal frequencies. They also used a proposed model in the parametric case study for a unimorph harvester and discussed modal electromechanical coupling and dependence of the electrical outputs on electrode locations. De Marqui Junior et al. [13] did research on an electromechanically coupled finite element (FE) plate model.
for predicting the electrical power output of piezoelectric energy harvester plates. They reviewed Generalized Hamilton’s principle for electroelastic bodies and derived the FE model based on the Kirchhoff plate assumptions as typical piezoelectric energy harvesters on thin structures. Ardila et al. [14] focused on mechanical energy harvesting using piezoelectric materials integrated into flexible substrates. They also considered a composite layer using piezoelectric nanostructures in order to evaluate its performance in a bending configuration. Li et al. [15] numerically studied an acoustic energy harvesting mechanism at low frequency (~200 Hz) using lead zirconate titanate (PZT) piezoelectric cantilever plates placed inside a quarter-wavelength straight-tube resonator and compared the results with experimental data. Dauksevicius et al. [16] presented an experimentally-verified multiphysics finite element model of a wideband vibration energy harvester with impact coupling that operates on the principle of frequency up-conversion. Maruccio et al. [17] developed a numerical approach for multiscale and multiphysics modeling of piezoelectric materials made of aligned arrays of polymeric nanofibers. Zhao et al. [18] presented electroelastic modeling, analytical and numerical solutions, and experimental validations of piezoelectric energy harvesting from broadband random vibrations. Stanton et al. [19] experimentally validated a first-principles based model for the nonlinear piezoelectric response of an electroelastic energy harvester. Diyanaa et al. [20] chose a unimorph piezoelectric energy harvester to harvest wideband mechanical energy. They used Euler-Bernoulli beam theory to derive the mathematical model. Chen et al. [21] presented the performance of a PZT diaphragm energy harvester. They showed that the energy from the harvester increases while its resonance frequency decreases when the pre-stress increasing. Zhou et al. [22] combined the electrical model with the piezoelectric constitutive equations of d15 mode and the single degree of freedom model to describe the energy harvesting performance of shear mode piezoelectric cantilever their proposed model is used to simulate the frequency dependence of the output peak voltage and power. They also discussed the effects of the material properties and geometrical dimensions on the energy harvesting performance to provide some useful guidelines to the design of piezoelectric energy harvesting devices. Aridogan et al. [23] presented coupled electroelastic modeling and experimental validations of broadband energy harvesting from structurally integrated piezoelectric patches on a rectangular thin plate. They developed a distributed-parameter electroelastic model for multiple patch based energy harvesters.

In this study, Flex PDE software has been used to solve, by using the finite element method (FEM), the transient response of axisymmetric plane strain in PZT materials in hollow cylinders when they are subjected to sinusoidal pressure force on the internal layer. PZT, or lead zirconate titanate is one of the world’s most widely used piezoelectric ceramic materials. Piezo ceramics are the preferred choice because they are physically strong, chemically inert and relatively inexpensive to manufacture. So investigating their mechanical and electrical behavior has great importance. For this aim, orthogonal expansion techniques and initial conditions as well as electrical boundary conditions for this method are conducted. Displacement, stresses, electric displacement and electric potential are obtained. The energy harvesting case is replicated, based on the Ding et al. study [1], to validate the results from Flex PDE. The results show that the present method is reliable for a hollow cylinder, with arbitrary thickness, which is subjected to arbitrary mechanical loads. Also, the transient responses of PZT-4 and PZT-5A piezoelectric materials for different boundary conditions have been modeled.

**CONSTITUTIVE EQUATIONS**

Piezoelectricity is described mathematically within a material's constitutive equation, which defines how the piezoelectric material's stress (σ), strain (s), charge-density displacement (D), and electric field (E) are coupled. The coupled constitutive equations can be taken empirically, as the linear combination of the pure mechanical or the pure electrical effect with the piezoelectric effect. The electro-mechanical property of this piezoelectric material can be described by the following constitutive equations.

\[ \nabla \cdot D_i = 0 \]  
(1)

where \( D \) is defined as:

\[ D_i = -\nabla \phi_i \]  
(2)

\[ \phi = -E \]  
(3)

In mathematical physics, the equations of motion describe the behavior of a physical system in terms of its motion as a function of time. More specifically, the equations of motion describe the behavior of a physical system as a set of mathematical functions in terms of dynamic variables. In this case time is considered as a dynamic variable and the equation of motion from elasticity theory is presented as follow.

\[ \nabla \sigma_{ij} + F = \rho \frac{\partial^2 u}{\partial t^2} \]  
(4)

where \( F \), \( \rho \), \( u \) and \( t \) are related to body force, density, displacement and time, respectively. For steady state analysis, the time variable is set to zero.

Equations (1) and (2) govern the distribution of electric field and stress in any direction of space. The strain of the system is defined as,

\[ s_{ij} = \frac{1}{2} \left( \nabla u_{ij} + (\nabla u_{ij})^T \right) \]  
(5)

Hence, the fundamental variables of this system are displacement and electric field potential, \( \phi \).

Constitutive relationships in piezoelectricity are presented in following form.

Stress: \[ \sigma_{ij} = C_{ijkl} s_{kl} + d_{ij} E_l \]  
(6)

Dielectric displacement: \[ D_l = d_{lj} E_l + \varepsilon E_j s_{ij} \]  
(7)
where C is equal to elastic coefficient. Equation (1) and (2) have to be solved simultaneously.

2D PIEZOELECTRIC EQUATIONS IN CYLINDRICAL COORDINATE

In order to model a piezoelectric material that has curvature form, cylindrical coordinates are used. In this coordinate system, \( u \) and \( v \) are the displacement in the radial (\( r \)) and circumferential (\( \theta \)) directions respectively. The component in the axial (\( z \)) direction is ignored. The modeling below follows from the Fesharaki paper [24]. The relations between strain and displacement in cylindrical coordinate are,

\[
\begin{align*}
\varepsilon_{rr} &= \frac{1}{2} \frac{\partial u}{\partial r} + \frac{u}{r} \\
\varepsilon_{\theta\theta} &= \frac{1}{2} \frac{\partial v}{\partial \theta} + \frac{v}{r} \\
\varepsilon_{r\theta} &= \frac{1}{2} \left( \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{1}{r} \frac{\partial v}{\partial r} \right)
\end{align*}
\]

The constitutive relationships for stress and dielectric displacement by ignoring axial direction, are considered as follows.

\[
\begin{align*}
T_{rr} &= C_{11} \varepsilon_{rr} + C_{12} \varepsilon_{\theta\theta} + d_{11} \frac{\partial \phi}{\partial r} + 11 \\
T_{\theta\theta} &= C_{22} \varepsilon_{rr} + C_{22} \varepsilon_{\theta\theta} + d_{22} \frac{\partial \phi}{\partial \theta} + 22 \\
T_{r\theta} &= 2C_{12} \varepsilon_{rr} + d_{31} \frac{\partial \phi}{\partial r}
\end{align*}
\]

The equations of motion in cylindrical coordinate system are considered as follows,

\[
\begin{align*}
\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} &= \frac{\partial^2 u}{\partial r^2} \\
\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{2T_{r\theta}}{r} &= \frac{\partial^2 \phi}{\partial r \partial \theta}
\end{align*}
\]

For steady state problems, the time variable would not be considered. The electric displacement field is defined by,

\[
\frac{\partial D_r}{\partial r} + \frac{1}{r} \frac{\partial D_{\theta}}{\partial \theta} + \frac{D_r}{r} = 0
\]

Equations (16) and (17) are coupled equations that need to be solved for piezoelectric with curvatures.

RESULTS AND DISCUSSION

In this study, a method has been developed by Flex PDE that uses FEM for solving the transient response of axisymmetric plane strain problem of PZT-4 and PZT-5A piezoelectric materials in the shape of hollow cylinders which is subjected to a sinusoidal time history pressure on the internal layer. Orthogonal expansion techniques and initial conditions as well as electrical boundary conditions for this method are conducted. Displacement, stresses, electric displacement and electric potential are obtained. The energy harvesting case is replicated based on the Ding et al. study [1] to validate the results from Flex PDE. The results show that the present method is reliable for the hollow cylinder with arbitrary thickness that is subjected to arbitrary mechanical loads. Also, the transient responses of PZT-4 and of PZT-5A piezoelectric materials for different boundary conditions are explored.

DYNAMIC RESPONSE OF HOLLOW CYLINDRICAL PIEZOELECTRIC MATERIAL

In this case, an axisymmetric plane strain problem based on the Ding et al. [1] study is considered. In the axisymmetric plain strain problem, the components of displacement and electric potential are reduced to \( u_r = u_r(r, z, t) \) and \( \phi = \phi(r, t) \) since all other components (circumferential and axial) are reduced to zero. Constitutive relationships are considered as equations (11), (12) and (14). So, the equation of motion and electric displacement field becomes,

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} \\
\frac{\partial^2 \phi}{\partial r \partial \theta} = 0
\]

Since the purpose of the study was to obtain a dynamic response, a time variable appears in the equation. In this study, Flex PDE was used to solve coupled equation (19) and (20) by FEM in order to replicate results from [25]. Boundary conditions are considered as follows; sinusoidal pressure acts on the internal surface and electric potential for the internal and external layer has been considered as zero Fig (1).
Table 1: Material properties of PZT-4 and PZT-5A

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>units</th>
<th>Material Type</th>
<th>PZT-4</th>
<th>PZT-5A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density</td>
<td>( \rho )</td>
<td>kg/m(^3)</td>
<td></td>
<td>7500</td>
<td>7750</td>
</tr>
<tr>
<td>Dielectric permittivity</td>
<td>( \varepsilon_{p13} )</td>
<td>( \frac{C^2}{N\cdot m^2} )</td>
<td></td>
<td>5.62 \text{x} 10^{-9}</td>
<td>13.27 \text{x} 10^{-9}</td>
</tr>
<tr>
<td>Anisotropic elastic constants</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( C_{11} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>139 \text{x} 10^9</td>
<td>151 \text{x} 10^9</td>
</tr>
<tr>
<td>( C_{12} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>77.8 \text{x} 10^9</td>
<td>98.4 \text{x} 10^9</td>
</tr>
<tr>
<td>( C_{13} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>74.3 \text{x} 10^9</td>
<td>96 \text{x} 10^9</td>
</tr>
<tr>
<td>( C_{22} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>139 \text{x} 10^9</td>
<td>151 \text{x} 10^9</td>
</tr>
<tr>
<td>( C_{23} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>74.3 \text{x} 10^9</td>
<td>96 \text{x} 10^9</td>
</tr>
<tr>
<td>( C_{33} )</td>
<td>Pa</td>
<td></td>
<td></td>
<td>115 \text{x} 10^8</td>
<td>124 \text{x} 10^9</td>
</tr>
<tr>
<td>Piezoelectric constant</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_{31} )</td>
<td>( \frac{C^2}{m^2} )</td>
<td></td>
<td></td>
<td>-5.2</td>
<td>-5.1</td>
</tr>
<tr>
<td>( d_{33} )</td>
<td>( \frac{C^2}{m^2} )</td>
<td></td>
<td></td>
<td>15.1</td>
<td>27</td>
</tr>
</tbody>
</table>

VALIDATION

Figure 2 shows a comparison between dynamic response from FEM results and results from the Ding paper. The presented dynamic responses are radial stress, \( \sigma_{rr} \), at \( r = 0.75 \text{ m} \) and Hoop stress, \( \sigma_{\theta\theta} \), at \( r = 0.5 \text{ m} \). Figure 2 shows good agreement between FlexPDE FEM results and the results from Ding’s study. The maximum response occurs at non-dimensional time \( t \approx 2.4 \). The differences between the magnitudes of response are because of different scales, since the FEM results are not non-dimensional analysis. The magnitudes of the radial stress and the hoop stress in this case are at \( 10^{11} \) and \( 10^{12} \text{ Pa} \) respectively.

![Figure 2: Radial stress at r=0.75 and Hoop stress at r=0.5, (a) from[1], (b) FlexPDE FEM results](image)

Figure 3 shows a comparison between electric field strength along a line on the cylinder and electric displacement at \( r = 0.75 \text{ m} \). The electric field strength along a line shows good agreement with the Ding study in terms of trend. At \( t = 0.1 \), the peak is around \( r = 0.63 \), and the peak moves along a radial direction until \( t = 0.5 \), the peak potential is around \( r = 0.82 \). Since analysis from the Ding results is non-dimensional, the magnitude of electric field strength is obviously different. The electric displacement at \( r = 0.75 \text{ m} \) from FEM result also shows good agreement of response trend with Ding results [1].

![Figure 3: Electric displacement at r=0.75 m and electric field strength from r=0.5 to r=1 m (a) results from Ding study [1] (b) FlexPDE FEM results](image)

Figure (4-a) shows deformation (magnified by 1011 times) caused by pressure. Since the pressure follows a sinusoidal function, the deformation changed direction. At \( t = 2 \text{ s} \), the direction of deformation is into the cylinder center, but at \( t=5.4 \), outward pressure on the internal surface caused deformation in the opposite direction. An electric potential will be generated in opposite signs due to the change in deformation direction. At \( t=2 \text{ s} \), the maximum positive electric potential is generated in the piezoelectric body by inward deformation. At this point, the negative electric potential was generated at the boundaries. But at \( t =5 \text{ s} \), the electric potential in the body away from the boundaries changed sign because of outward deformation of the body. This is consistent with the behavior of the piezoelectric medium.
MODELING PZT 4 AND 5A WITH DIFFERENT BOUNDARY CONDITIONS

In this part the same procedure has been followed for PZT-4 with different boundary conditions and with the same condition for PZT-5A with different material properties. This case is generally the actual boundary condition for energy harvesting. Sinusoidal pressure is applied on the internal surface and electric potential flux is considered as zero. The boundary conditions are listed as below:

\[ P_a(r_1, z, t) = -\sigma_z \sin(\omega_0 t) \]  \hspace{1cm} (22)
\[ P_b(r_2, z, t) = 0 \]  \hspace{1cm} (23)
\[ \frac{\partial \psi_r}{\partial n}(r_1, z, t) = 0 \hspace{1cm} (24) \]
\[ \frac{\partial \psi_r}{\partial n}(r_2, z, t) = 0 \]

where \( \sigma_0 \) is a prescribed constant stress and \( \sigma_0 = 1 \) has been taken for computation. Figure (5) shows the responses of \( \sigma_r \) at \( r \) equal to 0.5, 0.75 and 1 due to a random mechanical load shock. From the curves, it can be seen that the radial stress is increasing by increasing radius. Figure (6) gives the response of \( \sigma_0 \) at \( r \) equal to 0.5, 0.75 and 1 in the PZT-4 and PZT-5A materials. As is illustrated in the figures, the maximum values of dynamic hoop stress appear at the internal surface and are tensile stresses. The first peak value appears at time \( t = 1.8 \).

Figure 5: Radial stress at \( r = 0.5 \) & \( r = 0.75 \) & \( r = 1 \)

Figure 6: Hoop stress at \( r = 0.5 \) & \( r = 0.75 \) & \( r = 1 \)

Figure (7) illustrates the responses of \( D \) and the distributions of \( \phi \) at different positions \( r = 0.5, 0.75 \) and 1.0 in the PZT-4 and PZT-5A hollow cylinder subjected to a suddenly constant pressure on the internal surface. Since potential difference is directly proportionate to the current, from the figure it can be seen that the electric potential differences between internal and external surfaces can produce current in PZT materials.

Figure 7: Electric potential at \( r = 0.5 \) & \( r = 0.75 \) & \( r = 1 \)

CONCLUSION

In this study, Flex PDE software has been used to develop a method to solve, by using FEM, the transient response of the axisymmetric plane strain problem of PZT-4 and PZT-5A piezoelectric materials in hollow cylinders that are subjected to sinusoidal pressure force on the internal layer. The energy harvesting case is replicated based on Ding’s study to validate the results from Flex PDE. Displacement, stresses, electric displacement and electric potential are extracted. Comparison between the results has been made. The results show that the present method is reliable. Also, the transient responses of PZT-4 and PZT-5A piezoelectric materials for different boundary conditions are investigated. The utility of these materials as energy harvesters based on electric potential differences between internal and external layer has been discussed.

REFERENCES
